DOMINATION IN A DIGRAPH AND IN ITS REVERSE

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Abstract. Let D be a digraph. By $\gamma(D)$ we denote the domintaion number of D and by D^- we denote a digraph obtained by reversing all the arcs of D. In this paper we prove that for every $\delta \geq 3$ and $k \geq 1$ there exists a simple strongly connected δ-regular digraph $D_{\delta,k}$ such that $\gamma(D^-_{\delta,k}) - \gamma(D_{\delta,k}) = k$. Analogous theorem is obtained for total domination number provided that $\delta \geq 4$.

Key words and phrases. Domination number, total domination number, directed graph, reverse digraph, converse

1. Introduction and results

Let $D = (V(D), E(D))$ be a digraph. Then D is strongly connected if for every ordered pair u, v of its vertices there exists a directed $u - v$ path in D. If for every vertex v of D there are exactly δ arcs starting at v and exactly δ arcs terminating at v, then D is δ-regular. The reverse digraph D^- (which is sometimes called the converse of D) is obtained by reversing all the arcs of D. Let $v \in V(D)$. By $N(v)$ we denote the set of all neighbours of v, i.e., $N(v) = \{u;\ (v,u) \in E(D)\}$, while by $N[v]$ we denote the closed neighbourhood of v, i.e., $N[v] = N(v) \cup \{v\}$. A set S of vertices is a dominating set (total dominating set) if $\cup_{v\in S} N[v] = V(D)$ (if $\cup_{v\in S} N(v) = V(D)$). The minimum size of a dominating set (total dominating set) is the **domination number** $\gamma(D)$ (**total domination number** $\gamma_t(D)$) of D. Some authors use the notion "out-domination number" for $\gamma(D)$ and "in-domination number" for $\gamma(D^-)$, see e.g. [1].

The topic of domination belongs to most studied areas in graph theory. Problems of resource allocations and scheduling in networks are frequently formulated as domination problems on underlying (di)graphs, for terminology and survey of

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results see [6]. Comparing with graphs, there exists smaller number of results for domination in digraphs. The domination number in digraphs was introduced in [3]. If a digraph is antisymmetric, then both D and its reverse D^- are orientations of the same graph G. The relationship between domination numbers of different orientations of a graph was studied in [2]. A survey on domination in directed graphs is given in [4].

In [1] the authors prove the following theorem.

Theorem A. *For every digraph D of order $n \geq 2$ with no isolated vertices, the following bound is sharp:*

$$2 \leq \gamma(D) + \gamma(D^-) \leq \tfrac{4n}{3}.$$

While Theorem A bounds the sum of $\gamma(D)$ and $\gamma(D^-)$, we study their difference.

Let D be a weakly connected digraph on n vertices. Then its dominating number can be bounded by

$$1 \leq \gamma(D) \leq n - 1$$

where every value from the range $[1, n-1]$ is admissible, as can be shown by a suitable orientation of a star $K_{1,n-1}$. Better bounds can be expressed in terms of the maximal and minimal in- and out-degrees of vertices in D, see [7] and [4]. Anyway, the greatest difference of $\gamma(D^-) - \gamma(D)$ is $n-2$ as is shown by orientation of $K_{1,n-1}$ if we direct all the arcs from the center. The problem is that this digraph is not strongly connected and its total domination number is ∞. In the present note we show that the difference $\gamma(D^-) - \gamma(D)$ can not be bounded by a constant, even if we restrict to strongly connected regular digraphs. We present constructions of regular digraphs of given degree δ, where the difference between the (total) domination number of D^- and that of D is arbitrarily large. We prove the following two statements.

Theorem 1. *Let δ and k be integers, $\delta \geq 3$ and $k \geq 1$. Then there exists a simple strongly connected δ-regular digraph $D_{\delta,k}$ such that $\gamma(D^-_{\delta,k}) - \gamma(D_{\delta,k}) = k$.*

Theorem 2. *Let δ and k be integers, $\delta \geq 4$ and $k \geq 1$. Then there exists a simple strongly connected δ-regular digraph $C_{\delta,k}$ such that $\gamma_t(C^-_{\delta,k}) - \gamma_t(C_{\delta,k}) = k$.*

As regards small values of δ, the unique strongly connected 1-regular digraph is a directed cycle C. Since C^- is a digraph isomorphic to C, we have $\gamma(D) = \gamma(D^-)$ and $\gamma_t(D) = \gamma_t(D^-)$ for 1-regular strongly connected digraphs. However, the relation between $\gamma(D)$ and $\gamma(D^-)$ is not so obvious in the class of 2-regular strongly connected digraphs. Analogously, we do not know what is the relation between $\gamma_t(D)$ and $\gamma_t(D^-)$ in the class of 2-regular and 3-regular strongly connected digraphs. Therefore we pose the following problems:

Problem 1. Can be the difference $\gamma(D^-) - \gamma(D)$ arbitrarily large in the class of 2-regular strongly connected digraphs?

Problem 2. Can be the difference $\gamma_t(D^-) - \gamma_t(D)$ arbitrarily large in the classes of 2-regular and 3-regular strongly connected digraphs?

Since the ratio $\gamma(D^-)/\gamma(D)$ (as well as $\gamma_t(D^-)/\gamma_t(D)$) equals 7/6 in our proofs, we have another problem:

Problem 3. What is the greatest ratio $\gamma(D^-)/\gamma(D)$ (or $\gamma_t(D^-)/\gamma_t(D)$) if D is a δ-regular strongly connected digraph?

The proofs are postponed to the next section.

Let $D = (V(D), E(D))$ be a digraph (possibly with loops and multiple arcs) and let $\mathbb{A}$ be a group. Any mapping $\varphi : E(D) \to \mathbb{A}$ is called a voltage assignment and the value $\varphi(e)$ is the voltage on the arc e. Having voltage assignment on D, we can lift D to a larger digraph. The lifted digraph has vertex set $V(D) \times \mathbb{A}$ and there is an arc from (u, g) to (v, h) if and only if $e = (u, v)$ is an arc of D and $h = g \odot \varphi(e)$, where $\odot$ is the group operation in $\mathbb{A}$. As is a custom, we write u_g and v_h instead of (u, g) and (v, h), respectively. In this paper we use $\mathbb{A} = \mathbb{Z}_3$ only, so that $g, h \in \{0, 1, 2\}$. More general lifts are obtained by assigning permutations of $n = |\mathbb{A}|$ element set, say $\{0, 1, \dots, n-1\}$ to every arc of D. Denote by α_e the permutation assigned to the arc e. Then the lifted digraph has vertex set $V(D) \times \{0, 1, \dots, n-1\}$ and there is an arc from u_g to v_h if and only if $e = (u, v)$ is an arc of D and $h = \alpha_e(g)$. We mix these two types of voltage assignments in this paper, but as the underlying sets for both types of assignments will be identical (namely $\{0, 1, 2\}$), this will cause no problems. See [5] for more information about voltage assignments.

Let $x_0, x_1, \dots, x_{n-1}$ be vertices. By $(x_0, x_1, \dots, x_{n-1})^1$ we denote the arcs of a directed cycle $(x_0, x_1, \dots, x_{n-1})$, while by $(x_0, x_1, \dots, x_{n-1})^t$, $t \geq 2$, we denote arcs (x_i, x_{i+t}), $0 \leq i \leq n-1$ (the addition in subscript is modulo n).

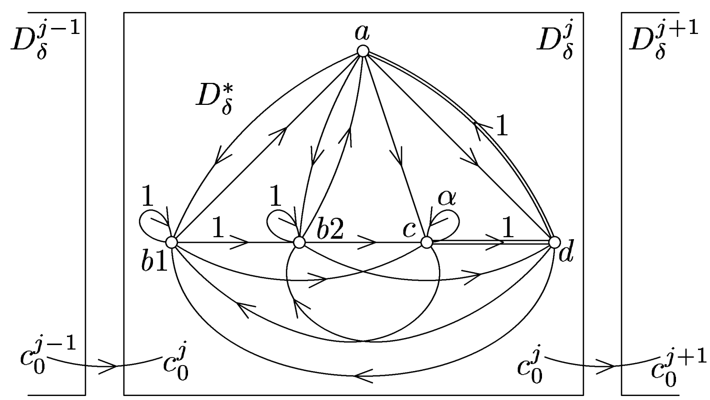

Figure 1. The digraph $D_{\delta,k}$ for $\delta = 4$.

Proof of Theorem 1. Let us denote by $D_\delta^* = (V(D_\delta^*), E(D_\delta^*))$ a δ-regular digraph on $\delta + 1$ vertices, where

$$\begin{aligned} V(D_\delta^*) &= \{a, b1, b2, \dots, b(\delta-2), c, d\}, \\ E(D_\delta^*) &= \{(a, c), (a, d), (d, a), (d, a), (c, c), (c, d)\} \\ &\quad \cup \{(a, bi), (bi, a), (bi, bi);\ 1 \leq i \leq \delta-2\} \\ &\quad \cup \cup_{j=1}^{\delta-2} (b1, b2, \dots b(\delta-2), c, d)^j. \end{aligned}$$

Observe that D_δ^* contains $\delta-1$ loops and two multiple arcs, namely (d, a) and (c, d). Now we assign voltages of $\mathbb{Z}_3$ to arcs of D_δ^*. All the arcs of D_δ^* receive voltage 0 except (bi, bi), $1 \leq i \leq \delta-2$, one of (c, d), one of (d, a) and $(b1, b2)$ (for the case $\delta \geq 4$), which receive voltage 1; and (c, c), which receives a permutation voltage $\alpha : (0)(1, 2)$. Now we construct the lifted digraph D_δ. This digraph contains no multiple arcs due to different voltages on parallel arcs, and it has only one loop, namely

(c_0, c_0). Analogously as D^*_δ, also D_δ is δ-regular and it is also strongly connected. Denote by D^j_δ a copy of D_δ with vertices $\{a^j_i, b1^j_i, \ldots, b(\delta-2)^j_i, c^j_i, d^j_i;\ 0 \le i \le 2\}$. Now take $2k$ copies $D^0_\delta, D^1_\delta, \ldots, D^{2k-1}_\delta$, remove from these copies the loops (c^j_0, c^j_0), $0 \le j \le 2k-1$, replace them by $(c^0_0, c^1_0, \ldots, c^{2k-1}_0)^1$, and denote the resulting digraph by $D_{\delta,k}$, see Figure 1 for the case $\delta = 4$. Then $D_{\delta,k}$ is simple and strongly connected δ-regular digraph. In the following we prove $\gamma(D_{\delta,k}) = 6k$ and $\gamma(D^-_{\delta,k}) = 7k$.

Since $D_{\delta,k}$ is δ-regular digraph on $2k \cdot 3(\delta+1)$ vertices, we have $\gamma(D_{\delta,k}) \ge 6k$. As $T = \cup_{j=0}^{2k-1}\{a^j_0, a^j_1, a^j_2\}$ is a dominating set of size $6k$, we have $\gamma(D_{\delta,k}) = 6k$.

Now consider $D^-_{\delta,k}$. In the following table we have for every vertex the list of its neighbours in $D^-_{\delta,k}$ (observe that if $\delta = 3$ then $b1 = b(\delta-2)$, i.e., the list of $b1^j_i$ terminates with d^j_i in that case; similarly the list of $b2^j_i$ terminates with d^j_i if $\delta = 4$ etc.).

$$
\begin{aligned}
a^j_i\ &: b1^j_i, b2^j_i, \ldots, b(\delta-2)^j_i, d^j_i, d^j_{i-1}\\
b1^j_i\ &: a^j_i, b1^j_{i-1}, d^j_i, c^j_i, b(\delta-2)^j_i, b(\delta-3)^j_i, \ldots b3^j_i\\
b2^j_i\ &: a^j_i, b2^j_{i-1}, b1^j_{i-1}, d^j_i, c^j_i, b(\delta-2)^j_i, b(\delta-3)^j_i, \ldots b4^j_i\\
&\ \ \vdots\\
bl^j_i\ &: a^j_i, bl^j_{i-1}, b(l-1)^j_i, b(l-2)^j_i, \ldots, b1^j_i, d^j_i, c^j_i, b(\delta-2)^j_i, b(\delta-3)^j_i, \ldots b(l+2)^j_i\\
&\ \ \vdots\\
b(\delta-2)^j_i\ &: a^j_i, b(\delta-2)^j_{i-1}, b(\delta-3)^j_i, b(\delta-4)^j_i, \ldots, b1^j_i, d^j_i\\
c^j_i\ &: a^j_i, b(\delta-2)^j_i, b(\delta-3)^j_i, \ldots, b1^j_i, c^*_{\alpha(i)}\\
d^j_i\ &: a^j_i, c^j_i, c^j_{i-1}, b(\delta-2)^j_i, b(\delta-3)^j_i, \ldots, b2^j_i
\end{aligned}
$$

We remark that the bottom indices are always modulo 3 and the upper indices are modulo $2k$. The vertex $c^*_{\alpha(i)}$ is c^{j-1}_0 if $i = 0$, it is c^j_2 if $i = 1$ and it is c^j_1 if $i = 2$.

Denote by S a dominating set in $D^-_{\delta,k}$ and denote $S^j = S \cap V(D^j_\delta)$. If $c^{j+1}_0 \in S^{j+1}$ then since c^{j+1}_0 has only $\delta - 1$ neighbours in $V(D^{j+1}_\delta)$ and since only c^{j+1}_0 can be dominated from outside S^{j+1}, we have $|S^{j+1}| \ge 4$. Now suppose that $c^{j+1}_0 \notin S^{j+1}$. We prove $|S^j| \ge 4$.

Since $c^{j+1}_0 \notin S^{j+1}$, all vertices of $V(D^j_\delta)$ are dominated by S^j. By contradiction, suppose that $|S^j| = 3$ and denote by x, y and z the three vertices of S^j. Moreover, denote by M the multiset consisting of $N[x]$, $N[y]$ and $N[z]$ in $D^-_{\delta,k}$. Then M contains every vertex of $V(D^j_\delta)$ exactly once. I.e., in M there is $a^j_.$ three times (with 3 different bottom indices 0, 1 and 2), also $bl^j_.$, $c^j_.$ and $d^j_.$ are 3 times each in M. Therefore S^j does not contain two $a^j_.$'s as then M would contain 4 times $d^j_.$. Analogously S^j does not contain two $bl^j_.$'s (due to four $bl^j_.$'s in M); S^j does not contain two $c^j_.$'s (due to four $c^j_.$'s in M, observe that $c^j_0 \notin S^j$ as shown above); and S^j does not contain two $d^j_.$'s (due to four $c^j_.$'s). Now suppose that we have in S^j one $bl^j_.$ and one $bt^j_.$ for $l < t$. Distinguish two cases:

Case 1: $t > l + 1$. Then $\delta \ge 5$. In $N(bf^j_.)$ there is missing exactly one of b's if $f < \delta-2$, namely $b(f+1)^j_.$, and $N(b(\delta-2)^j_.)$ contain all b's. Therefore in the multiset consisting of $N[bl^j_.]$ and $N[bt^j_.]$ we have three $bl^j_.$'s and three $bt^j_.$'s. Since there is either $bl^j_{\cdot}$ or $bt^j_{\cdot}$ in $N[v]$ for any $v \in V(D^j_\delta)$, the multiset M contains either four $bl^j_{\cdot}$'s or four $bt^j_{\cdot}$'s, a contradiction.

Case 2: $t = l+1$. Then $\delta \geq 4$. Analogously as in Case 1 one can see that there are three $bl^j_{\cdot}$'s in the multiset consisting of $N[bl^j_{\cdot}]$ and $N[bt^j_{\cdot}]$, so that the third vertex of S^j cannot have $bl^j_{\cdot}$ in its closed neighbourhood. That means that this third vertex is either $b(l-1)^j_{\cdot}$ if $l > 1$ or it is $d^j_{\cdot}$ if $l = 1$. Since $S^j = \{b(l-1)^j_{\cdot}, bl^j_{\cdot}, b(l+1)^j_{\cdot}\}$ was excluded in Case 1, we have $S^j = \{b1^j_{\cdot}, b2^j_{\cdot}, d^j_{\cdot}\}$. Suppose that $S^j = \{b1^j_i, b2^j_r, d^j_s\}$. Since $c^j_i \in N[b1^j_i]$ (recall that $\delta \geq 4$) and $c^j_s, c^j_{s-1} \in N[d^j_s]$, we have $s = i+2$ (recall that the arithmetics in bottom indices is considered modulo 3). Since $a^j_i \in N[b1^j_i]$, $a^j_{i+2} \in d^j_{i+2}$ and $a^j_r \in b2^j_r$, we have $r = i+1$. Therefore $S^j = \{b1^j_i, b2^j_{i+1}, d^j_{i+2}\}$. But then $b1^j_i$ occurs twice in M, a contradiction.

Thus, we have at most one of b's in S^j. If S^j contains $c^j_{\cdot}$ and $d^j_{\cdot}$, then either $c^j_o \in S^j$ in which case $|S^j| \geq 4$ as proved above, or there are four $c^j_{\cdot}$'s in M. Hence, S^j contains $a^j_{\cdot}$, $bl^j_{\cdot}$ for some l and either $c^j_{\cdot}$ or $d^j_{\cdot}$. However, if $S^k = \{a^j_{\cdot}, bl^j_{\cdot}, c^j_{\cdot}\}$ then M contains four $bl^j_{\cdot}$'s, while if $S^k = \{a^j_{\cdot}, bl^j_{\cdot}, d^j_{\cdot}\}$ then M contains four $d^j_{\cdot}$'s. Thus, we proved that $|S^j| \geq 4$ if $c^{j+1}_0 \notin S^{j+1}$.

Now $c^{j+1}_0 \in S^{j+1}$ gives $|S^{j+1}| \geq 4$ while $c^{j+1}_0 \notin S^{j+1}$ gives $|S^j| \geq 4$. This means that $|S^j \cup S^{j+1}| \geq 7$ and consequently $\gamma(D^-_{\delta,k}) = |S| \geq 7k$. It remains to find a dominating set of size $7k$ in $D^-_{\delta,k}$. Set $Q^j = \{a^j_0, a^j_1, c^j_2\}$. Then the only vertex of $V(D^j_\delta)$ which is not dominated by Q^j is c^j_0 (while d^j_0 is dominated "twice"). Therefore $R^{j+1} = \{a^{j+1}_0, a^{j+1}_1, c^{j+1}_2, c^{j+1}_0\}$ is a dominating set in D^{j+1}_δ. Consequently, $S = \cup_{j=0}^{k-1}(Q^{2j} \cup R^{2j+1})$ is a dominating set of size $7k$ in $D^-_{\delta,k}$, so that $\gamma(D^-_{\delta,k}) = 7k$. $\square$

Observe that $D_{\delta,k}$ can be obtained from D_δ by lifting in $\mathbb{Z}_{2k}$ if all the arcs of D_δ except (c_0, c_0) receive the voltage 0, while (c_0, c_0) receives voltage 1.

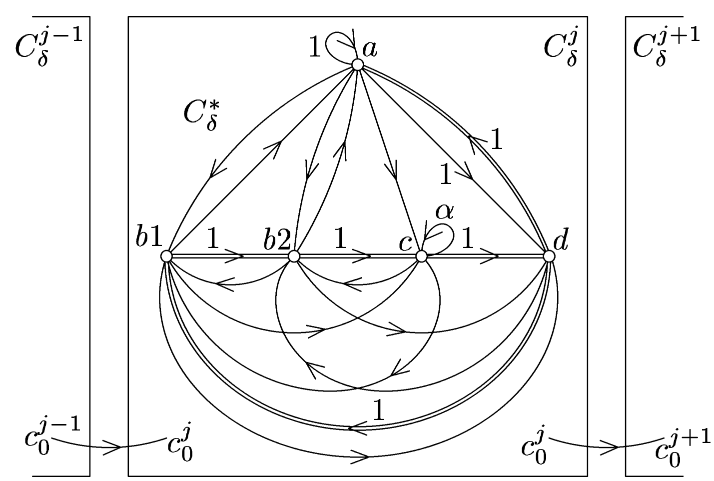

Figure 2. The digraph $C_{\delta,k}$ for $\delta = 5$.

Proof of Theorem 2. We construct $C_{\delta,k}$ similarly as was constructed $D_{\delta,k}$ in the proof of Theorem 1. Denote by $C^*_\delta = (V(C^*_\delta), E(C^*_\delta))$ a δ-regular digraph on δ

vertices, where

$$\begin{aligned}
V(C^*_\delta) &= \{a, b1, b2, \ldots, b(\delta-3), c, d\},\\
E(C^*_\delta) &= \{(a,a), (a,c), (a,d), (d,a), (d,a), (c,c)\}\\
&\quad \cup \{(a, bi), (bi, a);\ 1 \le i \le \delta-3\}\\
&\quad \cup (d, c, b(\delta-3), b(\delta-4), \ldots, b1)^1 - \{(d,c)\}\\
&\quad \cup (b1, b2, \ldots b(\delta-3), c, d)^1\\
&\quad \cup \cup_{j=1}^{\delta-3} (b1, b2, \ldots b(\delta-3), c, d)^j.
\end{aligned}$$

Then C^*_δ contains two loops, (a,a) and (c,c), and δ multiple arcs, namely (d,a) and $(b1, b2, \ldots, b(\delta-3), c, d)^1$. Now we assign voltages of $\mathbb{Z}_3$ to arcs of C^*_δ. All simple arcs of C^*_δ receive voltage 0 except (a,d), which receives voltage 1. Every pair of multiple arcs will receive voltages 0 and 1, the loop (a,a) receives voltage 1 and (c,c) receives permutation voltage $\alpha :\ (0)(1,2)$. The lifted digraph C_δ contains no multiple arcs and it has only one loop, namely (c_0, c_0). Further, C_δ is δ-regular and strongly connected. Denote by C^j_δ a copy of C_δ with vertices $\{a^j_i, b1^j_i, \ldots, b(\delta-3)^j_i, c^j_i, d^j_i;\ 0 \le i \le 2\}$. Take $2k$ copies $C^0_\delta, C^1_\delta, \ldots, C^{2k-1}_\delta$, remove from these copies the loops (c^j_0, c^j_0), $0 \le j \le 2k-1$, replace them by $(c^0_0, c^1_0, \ldots, c^{2k-1}_0)^1$, and denote the resulting digraph by $C_{\delta,k}$, see Figure 2 for the case $\delta = 5$. Then $C_{\delta,k}$ is simple and strongly connected δ-regular digraph. In the following we prove $\gamma_t(C_{\delta,k}) = 6k$ and $\gamma_t(C^-_{\delta,k}) = 7k$.

Since $C_{\delta,k}$ is δ-regular digraph on $2k \cdot 3\delta$ vertices, we have $\gamma_t(C_{\delta,k}) \ge 6k$. As $T = \cup_{j=0}^{2k-1}\{a^j_0, a^j_1, a^j_2\}$ is a total dominating set of size $6k$, we have $\gamma_t(C_{\delta,k}) = 6k$.

Now consider $C^-_{\delta,k}$. In the following table we have for every vertex the list of its neighbours in $C^-_{\delta,k}$

$$\begin{aligned}
a^j_i\ &: a^j_{i-1}, b1^j_i, b2^j_i, \ldots, b(\delta-3)^j_i, d^j_i, d^j_{i-1}\\
b1^j_i\ &: a^j_i, c^j_i, d^j_i, d^j_{i-1}, b2^j_i, b3^j_i, \ldots, b(\delta-3)^j_i\\
b2^j_i\ &: a^j_i, c^j_i, d^j_i, b1^j_i, b1^j_{i-1}, b3^j_i, b4^j_i, \ldots, b(\delta-3)^j_i\\
&\ \vdots\\
bl^j_i\ &: a^j_i, c^j_i, d^j_i, b1^j_i, b2^j_i, \ldots, b(l-1)^j_i, b(l-1)^j_{i-1}, b(l+1)^j_i, b(l+2)^j_i, \ldots, b(\delta-3)^j_i\\
&\ \vdots\\
b(\delta-3)^j_i\ &: a^j_i, c^j_i, d^j_i, b1^j_i, b2^j_i, \ldots, b(\delta-4)^j_i, b(\delta-4)^j_{i-1}\\
c^j_i\ &: a^j_i, b1^j_i, b2^j_i, \ldots, b(\delta-3)^j_i, b(\delta-3)^j_{i-1}, c^*_{\alpha(i)}\\
d^j_i\ &: a^j_{i-1}, c^j_i, c^j_{i-1}, b1^j_i, b2^j_i, \ldots, b(\delta-3)^j_i,
\end{aligned}$$

Analogously as in the proof of Theorem 1, the bottom indices are always modulo 3, the upper indices are modulo $2k$, and $c^*_{\alpha(i)}$ is c^{j-1}_0 if $i = 0$, it is c^j_2 if $i = 1$ and it is c^j_1 if $i = 2$.

Denote by S a total dominating set in $C^-_{\delta,k}$ and denote $S^j = S \cap V(C^j_\delta)$. If $c^{j+1}_0 \in S^{j+1}$ then since c^{j+1}_0 has only $\delta - 1$ neighbours in $V(C^{j+1}_\delta)$ and since only

c_0^{j+1} can be dominated from outside S^{j+1}, we have $|S^{j+1}| \geq 4$. Now suppose that $c_0^{j+1} \notin S^{j+1}$. We prove $|S^j| \geq 4$.

Since $c_0^{j+1} \notin S^{j+1}$, all vertices of $V(C_\delta^j)$ are dominated by S^j. By contradiction, suppose that $|S^j| = 3$ and denote by x, y and z the three vertices of S^j. Denote by M the multiset consisting of $N(x)$, $N(y)$ and $N(z)$ in $C_{\delta,k}^-$. Then M contains every vertex of $V(C_\delta^j)$ exactly once. Therefore S^j does not contain two $a_\cdot^j$'s as then M would contain 4 times $d_\cdot^j$, S^j does not contain two $bl_\cdot^j$'s (due to four $b(l{-}1)_\cdot^j$'s if $l > 1$ and four $d_\cdot^j$'s if $l = 1$); S^j does not contain two $c_\cdot^j$'s (due to four $b(\delta{-}3)_\cdot^j$'s); and S^j does not contain two $d_\cdot^j$'s (due to four $c_\cdot^j$'s). Now suppose that we have in S^j one $bl_\cdot^j$ and one $bt_\cdot^j$ for $l < t$. Distinguish two cases:

Case 1: $l > 1$. In $N(bf_\cdot^j)$ there is missing exactly one of b's, namely $bf_\cdot^j$. Since $b(l{-}1)_\cdot^j$ is twice in $N(bl_\cdot^j)$, in the multiset consisting of $N(bl_\cdot^j)$ and $N(bt_\cdot^j)$ we have three $b(l{-}1)_\cdot^j$'s (recall that $t > l$). That means that the third element of S^j is $b(l{-}1)_\cdot^j$. Now if $l = 2$ then there are four $d_\cdot^j$'s in M, a contradiction. On the other hand, if $l > 2$ then there are four $b(l{-}2)_\cdot^j$'s in M, a contradiction.

Case 2: $l = 1$. Then the multiset consisting of $N(b1_\cdot^j)$ and $N(bt_\cdot^j)$ contains three $d_\cdot^j$'s, which means that the third element of S^j is either $c_\cdot^j$ or $d_\cdot^j$. If $S^j = \{b1_\cdot^j, bt_\cdot^j, d_\cdot^j\}$ then we have four $c_\cdot^j$'s in M, a contradiction. Hence, suppose that $S^j = \{b1_\cdot^j, bt_\cdot^j, c_\cdot^j\}$. Since M contains four $b(\delta{-}3)_\cdot^j$'s if $t \neq \delta{-}3$, we have $t = \delta{-}3$. Since M contains four $b(\delta{-}4)_\cdot^j$'s if $1 < \delta{-}4$, we have $\delta = 5$. Thus, $\delta{-}3 = 2$ and $S^j = \{b1_i^j, b2_r^j, c_s^j\}$ for some i, r and s. Since $d_i^j, d_{i-1}^j \in N(b1_i^j)$ and $d_r^j \in N(b2_r^j)$, we have $r = i + 1$ (recall that the arithmetics in bottom indices is considered modulo 3). Since $b2_i^j \in N(b1_i^j)$ and $b2_s^j, b2_{s-1}^j \in N(c_s)$, we have $s = i + 2$. But then $S^j = \{b1_i^j, b2_{i+1}^j, c_{i+2}^j\}$ and $c_{i+2}^j \notin M$, a contradiction.

Thus, we have at most one of b's in S^j. If S^j contains $c_\cdot^j$ and $d_\cdot^j$, then since neither $N(c_\cdot^j)$ nor $N(d_\cdot^j)$ contain d's, there are at most two $d_\cdot^j$'s in M. Hence, S^j contains $a_\cdot^j$, $bl_\cdot^j$ for some l and either $c_\cdot^j$ or $d_\cdot^j$. However, if $S^k = \{a_\cdot^j, bl_\cdot^j, c_\cdot^j\}$ then M contains only two $c_\cdot^j$'s, while if $S^k = \{a_\cdot^j, bl_\cdot^j, d_\cdot^j\}$ then M contains only two $bl_\cdot^j$'s. Thus, we proved that $|S^j| \geq 4$ if $c_0^{j+1} \notin S^{j+1}$.

Analogously as in the proof of Theorem 1 we conclude $|S^j \cup S^{j+1}| \geq 7$ and consequently $\gamma_t(C_{\delta,k}^-) = |S| \geq 7k$. It remains to find a total dominating set of size $7k$ in $C_{\delta,k}^-$. Set $Q^j = \{a_0^j, a_1^j, d_2^j\}$. Then the only vertex of $V(C_\delta^j)$ which is not dominated by Q^j is c_0^j (while d_0^j is dominated "twice"). Therefore $R^{j+1} = \{a_0^{j+1}, a_1^{j+1}, d_2^{j+1}, c_0^{j+1}\}$ is a total dominating set in C_δ^{j+1}. Consequently, $S = \cup_{j=0}^{k-1}(Q^{2j} \cup R^{2j+1})$ is a total dominating set of size $7k$ in $C_{\delta,k}^-$, so that $\gamma_t(C_{\delta,k}^-) = 7k$. □

Analogously as $D_{\delta,k}$, also $C_{\delta,k}$ can be obtained from C_δ by lifting in $\mathbb{Z}_{2k}$.

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