Biembeddings of Abelian groups

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Abstract

We prove that, with the single exception of the 2-group C_2^2, the Cayley table of each Abelian group appears in a face 2-colourable triangular embedding of a complete regular tripartite graph in an orientable surface.

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1 Background

A biembedding of two Latin squares of order n is equivalent to a face 2-colourable triangular embedding of a complete regular tripartite graph $K_{n,n,n}$ in which the faces of each colour class generate the Latin squares. In this paper we determine which Latin squares formed from the Cayley tables of Abelian groups appear in such biembeddings.

In [4] a recursive construction was presented for biembeddings of Latin squares. This construction was used in that paper to prove that, for $i \neq 2$, the Cayley table of the Abelian 2-group C_2^i appears in a biembedding. It was also conjectured that, with the single exception of the group C_2^2, the Cayley table of each Abelian group appears in a biembedding. In the current paper we give a proof of this conjecture. The difficulty in proving the result is due to the facts that the Cayley table of C_2^2 appears in no biembedding, while the unique biembeddings in which the Cayley tables of C_2 and C_4 appear are with copies of themselves, and both these squares lack transversals.

For general background material on topological embeddings, we refer the reader to [5] and [6]. Our embeddings will always be 2-cell embeddings in closed connected 2-manifolds without a boundary. A graph embedding is *face 2-colourable* if the faces may be coloured in such a way that any two faces with a common boundary edge receive different colours. It was shown in [2] that a triangular embedding of $K_{n,n,n}$ is face 2-colourable if and only if the supporting surface is orientable, and the surface is therefore a sphere with an appropriate number of handles.

A face 2-colourable triangular embedding of $K_{n,n,n}$ determines two *transversal designs*, $\mathrm{TD}(3,n)$, one for each colour class. Such a design comprises an ordered triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into three disjoint sets (the *groups*) each of cardinality n, and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every unordered pair of elements from V is either contained in precisely one triple or one group, but not both. The vertices of the embedded graph $K_{n,n,n}$ form the points of each design, the tripartition determines the groups, and the faces in each colour class form the triples of each design.

The connection with Latin squares is that a $\mathrm{TD}(3,n)$ determines a Latin square of order n by assigning the three groups of the design as labels for the rows, columns and entries (in any one of six possible orders) of the Latin square. Conversely any Latin square of order n determines a $\mathrm{TD}(3,n)$. Two Latin squares are said to be in the same *main class* or *paratopic* if the corresponding $\mathrm{TD}(3,n)$s are isomorphic. Thus a face 2-colourable triangular embedding of $K_{n,n,n}$ may be considered as a biembedding of two $\mathrm{TD}(3,n)$s or, equivalently, two Latin squares. To be precise, we say that two Latin squares of order n are *biembeddable* in a surface if there is a face 2-colourable triangular embedding of $K_{n,n,n}$ in which the face sets forming the two colour classes give paratopic copies of the two squares.

Given a Latin square L of order n, we may use the notation $k = L(i, j)$ to denote that entry k appears in row i column j of L; alternatively we may write $(i, j, k) \in L$. In this latter form, the triples of any Latin square will always be specified in (row, column, entry) order. Note however that in a biembedding of two Latin squares, the vertices of faces forming one colour class will appear clockwise in the cyclic order (row, column, entry), while those forming the other will appear anticlockwise if taken in the same cyclic order. A *parallel class* of triples in a $\mathrm{TD}(3, n)$ is a set of triples in which each point of the design appears precisely once. Such a parallel class is equivalent to a transversal in a corresponding Latin square.

For Latin squares A and B of order n with common sets of row labels, of column labels, and of entries, we will write $A \bowtie B$ (to be read as A *biembeds with* B *without relabelling*), if the particular realizations of A and B form an embedding in a surface; that is to say that the triangles formed by the (row, column, entry) triples of A and B may be sewn together along their common edges to form the surface. As a matter of terminology, we will refer to vertices as row, column or entry vertices, so that a triple $(a, b, c) \in A$ gives a face with row vertex a, column vertex b and entry vertex c. In order to verify that $A \bowtie B$, it is necessary to check that the sewing operation generates a genuine surface and not a pseudosurface. This can be done by checking that the rotation at each vertex is a single cycle of length $2n$ rather than a set of shorter cycles. With a slight abuse of notation we also use $A \bowtie B$ to denote the actual embedding itself. Furthermore, if B is known to have a transversal, we will add a + sign and write $A \bowtie B+$. We will also identify a group G with its Cayley table, so that we may write $G \bowtie H$, meaning that the Latin square formed by a Cayley table of G biembeds with the Latin square H.

2 The theorem

Theorem 2.1 *Suppose that G is an Abelian group and that $G \neq C_2^2$. Then $G \bowtie H$ for some Latin square H. There is no H for which $C_2^2 \bowtie H$.*

The proof of this result follows from the construction given in [4], a known result concerning regular embeddings, and four additional lemmas. We start by citing the earlier results.

Theorem 2.2 [4] *Suppose that $L \bowtie L'$, where L and L' are of order n and have row, column and entry labels $\{0, 1, \ldots, n-1\}$. Suppose also that $Q \bowtie Q'+$, where Q and Q' are of order m and have row, column and entry labels $\{0, 1, \ldots, m-1\}$, and that the square Q' has a transversal $\mathcal{T}$. Define squares $Q(L)$ and $Q'(L, \mathcal{T}, L')$ by*

$$\begin{aligned} Q(L)(nu+i, nv+j) &= nQ(u,v) + L(i,j), \\ Q'(L, \mathcal{T}, L')(nu+i, nv+j) &= nQ'(u,v) + k, \end{aligned}$$

for $0 \le u, v \le m-1$ and $0 \le i, j \le n-1$, where

$$k = \begin{cases} L(i,j) & \text{if } (u,v,w) \notin \mathcal{T} \text{ for any } w, \\ L'(i,j) & \text{if there exists } w \text{ such that } (u,v,w) \in \mathcal{T}, \end{cases}$$

Then $Q(L)$ and $Q'(L, \mathcal{T}, L')$ are Latin squares of order mn with row, column and entry labels $\{0, 1, \ldots, mn-1\}$, and $Q(L) \bowtie Q'(L, \mathcal{T}, L')$.

The square $Q(L)$ is partitioned into $n \times n$ subsquares which are just relabelled copies of L. The square $Q'(L, \mathcal{T}, L')$ has a similar structure but the subsquares corresponding to the transversal $\mathcal{T}$ are relabelled copies of L'. Note that if L' has a transversal, then among the relabelled copies of L' one can find a transversal in $Q'(L, \mathcal{T}, L')$. This feature facilitates re-application of the construction. Note also that if Q and L are groups then $Q(L)$ is a Cayley table for the group $Q \times L$.

As an application of Theorem 2.2, it was shown in [4] that all Abelian 2-groups, apart from C_2^2, appear in biembeddings.

Theorem 2.3 [4] *For every $i \ge 1$, $i \ne 2$, there is a Latin square A_i such that $C_2^i \bowtie A_i$. Moreover, if $i > 2$ then the square A_i may be taken to contain a transversal. There is no A_2 such that $C_2^2 \bowtie A_2$.*

The next result, the present form of which is taken from [2], asserts the existence of a biembedding of each cyclic group C_t. To explain the terminology we digress slightly. We will say that an embedding of a graph G is *regular* or *flag-transitive* if for every two *flags*, i.e. triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with the vertex v_i and the face f_i, there exists an automorphism of the embedding which maps v_1 to v_2, e_1 to e_2 and f_1 to f_2. This definition of regularity requires the admission of automorphisms that reverse the orientation of an orientable surface, although some authors require global orientation to be preserved. We refer the reader to [1, p.36] for further discussion of the terminology. When viewed as a face 2-colourable triangular embedding of $K_{t,t,t}$, the biembedding described in Theorem 2.4 is regular and, as shown in [3], is (up to isomorphism) the unique biembedding of C_t with this regularity property. Accordingly, we will refer to it as the regular biembedding of C_t.

Theorem 2.4 [2] *If the Latin squares C_t and C'_t are defined by $C_t(i,j) = i + j \mod t$ and $C'_t(i,j) = i + j + 1 \mod t$, then $C_t \bowtie C'_t$. Moreover, if t is odd then C'_t has a transversal $\mathcal{T} = \{(i, i, 2i+1) : i \in \mathbb{Z}_t\}$.*

We next state and prove the four lemmas that will enable us to complete the proof of Theorem 2.1.

Lemma 2.1 *If $i \geq 3$ then $C_{2^i} \bowtie H_i+$ for some H_i having a transversal.*

Proof. We deal first with the case $i = 3$. It is easy to check that the square C_8 biembeds with

$$H_3 = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 7 & 2 & 3 & 0 & 1 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 1 & 6 & 7 & 0 & 2 \\ 2 & 1 & 5 & 6 & 4 & 7 & 0 & 2 & 3 \\ 3 & 6 & 3 & 7 & 5 & 0 & 2 & 4 & 1 \\ 4 & 5 & 7 & 0 & 6 & 2 & 3 & 1 & 4 \\ 5 & 2 & 0 & 4 & 7 & 3 & 1 & 6 & 5 \\ 6 & 0 & 6 & 1 & 2 & 4 & 5 & 3 & 7 \\ 7 & 4 & 1 & 2 & 3 & 5 & 6 & 7 & 0 \end{array}$$

Note that H_3 has a transversal (shown highlighted) on the leading diagonal.

For $i > 3$, put $n = 2^i$. To form H_i, start with C'_n as defined above but alter 16 entries to give the following triples:

$$\begin{array}{cccc} (0, 0, \frac{n}{2}+1), & (0, \frac{n}{4}, \frac{3n}{4}+1), & (0, \frac{n}{2}, 1), & (0, \frac{3n}{4}, \frac{n}{4}+1), \\ (\frac{n}{4}, 0, \frac{3n}{4}+1), & (\frac{n}{4}, \frac{n}{4}, 1), & (\frac{n}{4}, \frac{n}{2}, \frac{n}{4}+1), & (\frac{n}{4}, \frac{3n}{4}, \frac{n}{2}+1), \\ (\frac{n}{2}, 0, 1), & (\frac{n}{2}, \frac{n}{4}, \frac{n}{4}+1), & (\frac{n}{2}, \frac{n}{2}, \frac{n}{2}+1), & (\frac{n}{2}, \frac{3n}{4}, \frac{3n}{4}+1), \\ (\frac{3n}{4}, 0, \frac{n}{4}+1), & (\frac{3n}{4}, \frac{n}{4}, \frac{n}{2}+1), & (\frac{3n}{4}, \frac{n}{2}, \frac{3n}{4}+1), & (\frac{3n}{4}, \frac{3n}{4}, 1). \end{array}$$

Since $C_n \bowtie C'_n$, it follows that in the embedding of C_n with H_i the rotations at row vertices, other than those corresponding to rows $0, \frac{n}{4}, \frac{n}{2}$ and $\frac{3n}{4}$, will be cycles of length $2n$. The same goes for column vertices, while for entry vertices the only possible exceptions are for the entries $1, \frac{n}{4}+1, \frac{n}{2}+1$ and $\frac{3n}{4}+1$. The rotation at a row vertex alternates column and entry vertices, so to prove that the rotation at the row vertex 0 is also a single cycle of length $2n$, it suffices to list these entry vertices in the order in which they appear and to verify that they form a single cycle of length n. The sequence(s) of entry vertices around this row vertex is (are) determined by the following permutation given in two-line form, where the top line is row 0 of C_n and the bottom line is row 0 of H_i.

$$\left(\begin{array}{cccccccccccc} 0 & 1 & 2 & \dots & \frac{n}{4}-1 & \frac{n}{4} & \frac{n}{4}+1 & \dots & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+1 & \dots \\ \frac{n}{2}+1 & 2 & 3 & \dots & \frac{n}{4} & \frac{3n}{4}+1 & \frac{n}{4}+2 & \dots & \frac{n}{2} & 1 & \frac{n}{2}+2 & \dots \end{array} \right.$$
$$\left. \begin{array}{cccccc} \dots & \frac{3n}{4}-1 & \frac{3n}{4} & \frac{3n}{4}+1 & \dots & n-1 \\ \dots & \frac{3n}{4} & \frac{n}{4}+1 & \frac{3n}{4}+2 & \dots & 0 \end{array} \right).$$

This gives

$$(1, 2, 3, \dots, \tfrac{n}{4}-1, \tfrac{n}{4}, \tfrac{3n}{4}+1, \tfrac{3n}{4}+2, \dots, n-1, 0, \tfrac{n}{2}+1, \tfrac{n}{2}+2, \dots$$
$$\dots, \tfrac{3n}{4}-1, \tfrac{3n}{4}, \tfrac{n}{4}+1, \tfrac{n}{4}+2, \dots, \tfrac{n}{2}-1, \tfrac{n}{2})$$

which is a single cycle of length n. The same cycle will be obtained for rows $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{3n}{4}$ because these rows of C_n and H_i are just cyclic shifts of row 0 by $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{3n}{4}$ respectively. Furthermore, the squares are symmetric, so the same cycle will be obtained for the columns labelled $0, \frac{n}{4}, \frac{n}{2}$ and $\frac{3n}{4}$.

Finally we consider the rotations at the four exceptional entry vertices. Each such rotation alternates row and column vertices, and if the row vertices form a cycle of length n then the rotation will form a single cycle of length $2n$. We start by considering entry 1, and constructing a permutation representing the sequence(s) of row vertices around this entry vertex. In two-line format this is given by taking the top line to be the list, in column order, of the row labels corresponding to the entry 1 in C_n, and the bottom line is formed similarly from H_i. It may help the reader for us to note that, with arithmetic modulo n, column j gives rise to a pair of the form ${}^{1-j}_{\ x}$ in the permutation. The permutation is as follows.

$$\left(\begin{array}{cccccccccccc} 1 & 0 & n-1 & \dots & \frac{3n}{4}+2 & \frac{3n}{4}+1 & \frac{3n}{4} & \dots & \frac{n}{2}+2 & \frac{n}{2}+1 & \frac{n}{2} & \dots \\ \frac{n}{2} & n-1 & n-2 & \dots & \frac{3n}{4}+1 & \frac{n}{4} & \frac{3n}{4}-1 & \dots & \frac{n}{2}+1 & 0 & \frac{n}{2}-1 & \dots \end{array} \right.$$
$$\left. \begin{array}{cccccc} \dots & \frac{n}{4}+2 & \frac{n}{4}+1 & \frac{n}{4} & \dots & 2 \\ \dots & \frac{n}{4}+1 & \frac{3n}{4} & \frac{n}{4}-1 & \dots & 1 \end{array} \right).$$

This gives

$$\left(0, n-1, n-2, \dots, \tfrac{3n}{4}+2, \tfrac{3n}{4}+1, \tfrac{n}{4}, \tfrac{n}{4}-1, \dots, 2, 1, \tfrac{n}{2}, \tfrac{n}{2}-1, \dots\right.$$
$$\left.\dots, \tfrac{n}{4}+2, \tfrac{n}{4}+1, \tfrac{3n}{4}, \tfrac{3n}{4}-1, \dots, \tfrac{n}{2}+2, \tfrac{n}{2}+1\right)$$

which is a single cycle of length n (in fact the inverse of the row and column cycle). By the same argument as for the rows, the same cycle will be obtained for the entries $\frac{n}{4}+1$, $\frac{n}{2}+1$ and $\frac{3n}{4}+1$. It therefore follows that $C_n \bowtie H_i$.

All that remains is to identify a transversal in H_i. Noting the assumption that $n \geq 16$, this is given by the triples

$$(x, x+2, 2x+3) \text{ for } x = 0, 1, \dots, \tfrac{n}{2}-3, \text{ except for } x = \tfrac{n}{4}-1,$$
$$(x, x-1, 2x) \text{ for } x = \tfrac{n}{2}+2, \tfrac{n}{2}+3, \dots, n-1, \text{ except for } x = \tfrac{3n}{4},$$
$$(\tfrac{n}{4}-1, \tfrac{3n}{4}-1, n-1), (\tfrac{n}{2}-2, 1, \tfrac{n}{2}), (\tfrac{n}{2}-1, \tfrac{n}{2}, 0), (\tfrac{n}{2}, 0, 1),$$
$$(\tfrac{n}{2}+1, n-1, \tfrac{n}{2}+1), (\tfrac{3n}{4}, \tfrac{n}{4}+1, 2). \qquad \square$$

Lemma 2.2 *If $i \geq 2$ then $C_4^i \bowtie H_i+$ for some H_i having a transversal.*

Proof. We show that $C_4^2 \bowtie H_2+$ and $C_4^3 \bowtie H_3+$ for some H_2 and H_3, each having a transversal. The result will then follow by use of Theorem 2.2 since, for $i \geq 4$, we have $C_4^i = C_4^2 \times C_4^2 \times \cdots \times C_4^2 \times C_4^j$, where $j = 2$ or 3. We may take

$H_2 =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	7	4	5	14	0	1	2	3	15	12	13	$\boxed{6}$	8	9	10	11
1	6	7	12	5	1	2	3	0	14	15	$\boxed{4}$	13	9	10	11	8
2	5	6	7	4	10	3	0	1	13	14	15	12	$\boxed{2}$	11	8	9
3	4	5	6	7	3	8	1	2	12	13	14	15	11	$\boxed{0}$	9	10
4	2	11	0	1	5	6	7	4	10	$\boxed{3}$	8	9	13	14	15	12
5	0	1	2	3	7	4	13	6	8	9	10	11	15	12	$\boxed{5}$	14
6	3	0	1	2	4	5	6	15	11	8	9	10	12	13	14	$\boxed{7}$
7	9	2	3	0	6	7	4	5	$\boxed{1}$	10	11	8	14	15	12	13
8	15	12	13	6	8	9	$\boxed{10}$	11	7	4	5	14	0	1	2	3
9	14	$\boxed{15}$	4	13	9	10	11	8	6	7	12	5	1	2	3	0
10	13	14	15	12	2	11	8	$\boxed{9}$	5	6	7	4	10	3	0	1
11	$\boxed{12}$	13	14	15	11	0	9	10	4	5	6	7	3	8	1	2
12	10	3	$\boxed{8}$	9	13	14	15	12	2	11	0	1	5	6	7	4
13	8	9	10	$\boxed{11}$	15	12	5	14	0	1	2	3	7	4	13	6
14	11	8	9	10	12	$\boxed{13}$	14	7	3	0	1	2	4	5	6	15
15	1	10	11	8	$\boxed{14}$	15	12	13	9	2	3	0	6	7	4	5

To verify that C_4^2 biembeds with H_2 the reader should check the rotations at each of the row, column and entry vertices. The square H_2 has two disjoint transversals, one highlighted, $\mathcal{T}_2$, and one boxed, $\mathcal{T}_2'$. The former is given by $\mathcal{T}_2 = \{(0,3,14),\ (1,2,12),\ (2,4,10),\ (3,5,8),\ (4,1,11),\ (5,6,13),\ (6,7,15),\ (7,0,9),\ (8,14,2),\ (9,9,7),\ (10,15,1),\ (11,8,4),\ (12,10,0),\ (13,11,3),\ (14,13,5),\ (15,12,6)\}$. It follows from Theorem 2.2 that $C_4^3 = C_4^2(C_4)$ biembeds with $H_3 = H_2(C_4, \mathcal{T}_2, C_4')$. These squares are both of order 64, and consequently, too large to display here. Nevertheless, the structure of H_3 should be clear and it has a transversal, $\mathcal{T}_3$, obtained from $\mathcal{T}_2$ and $\mathcal{T}_2'$ and given by the 64 triples shown in Table 1. This completes the proof of the lemma. $\square$

$(0,12,57)$,	$(1,13,59)$,	$(2,46,24)$,	$(3,47,26)$,
$(4,8,49)$,	$(5,9,51)$,	$(6,42,16)$,	$(7,43,18)$,
$(8,16,41)$,	$(9,17,43)$,	$(10,50,8)$,	$(11,51,10)$,
$(12,20,33)$,	$(13,21,35)$,	$(14,54,0)$,	$(15,55,2)$,
$(16,4,45)$,	$(17,5,47)$,	$(18,38,12)$,	$(19,39,14)$,
$(20,24,53)$,	$(21,25,55)$,	$(22,58,20)$,	$(23,59,22)$,
$(24,28,61)$,	$(25,29,63)$,	$(26,62,28)$,	$(27,63,30)$,
$(28,0,37)$,	$(29,1,39)$,	$(30,34,4)$,	$(31,35,6)$,
$(32,56,9)$,	$(33,57,11)$,	$(34,26,40)$,	$(35,27,42)$,
$(36,36,29)$,	$(37,37,31)$,	$(38,6,60)$,	$(39,7,62)$,
$(40,60,5)$,	$(41,61,7)$,	$(42,30,36)$,	$(43,31,38)$,
$(44,32,17)$,	$(45,33,19)$,	$(46,2,48)$,	$(47,3,50)$,
$(48,40,1)$,	$(49,41,3)$,	$(50,10,32)$,	$(51,11,34)$,
$(52,44,13)$,	$(53,45,15)$,	$(54,14,44)$,	$(55,15,46)$,
$(56,52,21)$,	$(57,53,23)$,	$(58,22,52)$,	$(59,23,54)$,
$(60,48,25)$,	$(61,49,27)$,	$(62,18,56)$,	$(63,19,58)$.

Table 1. The transversal $\mathcal{T}_3$ in H_3.

Lemma 2.3 *If $i \geq 2$ and $n = 2^i$, then $(C_2 \times C_n) \bowtie H_n+$ for some H_n having a transversal.*

Proof. The proof is by direct construction. The square H_n is a copy of the Cayley table of the dihedral group D_n. We give standard forms for the Cayley tables of $C_2 \times C_n$ and D_n, and then give three permutations that are applied respectively to the entries, column labels and row labels of D_n in order to form H_n. In the next paragraph it is shown that D_n, and hence H_n, has a transversal. Rather more tedious is the proof that the squares $C_2 \times C_n$ and H_n form a biembedding; we do this by examining the rotation at each vertex. In our squares of order $2n$, the row labels, column labels and entries will be taken as $0, 1, \ldots, n-1, 0', 1', \ldots, n-1'$, where $n-1'$ is written for $(n-1)'$ to save on excessive use of brackets; a similar gloss will be applied to other compound terms. All arithmetic encountered is to be taken in $\mathbb{Z}_n$. Our standard form for D_n is shown in Figure 1.

	0	1	2	$\ldots$	$n-1$	$0'$	$1'$	$2'$	$\ldots$	$n-1'$
0	0	1	2	$\ldots$	$n-1$	$0'$	$1'$	$2'$	$\ldots$	$n-1'$
1	1	2	3	$\ldots$	0	$1'$	$2'$	$3'$	$\ldots$	$0'$
2	2	3	4	$\ldots$	1	$2'$	$3'$	$4'$	$\ldots$	$1'$
$\vdots$				$\vdots$					$\vdots$	
$n-1$	$n-1$	0	1	$\ldots$	$n-2$	$n-1'$	$0'$	$1'$	$\ldots$	$n-2'$
$0'$	$0'$	$n-1'$	$n-2'$	$\ldots$	$1'$	0	$n-1$	$n-2$	$\ldots$	1
$1'$	$1'$	$0'$	$n-1'$	$\ldots$	$2'$	1	0	$n-1$	$\ldots$	2
$2'$	$2'$	$1'$	$0'$	$\ldots$	$3'$	2	1	0	$\ldots$	3
$\vdots$				$\vdots$					$\vdots$	
$n-1'$	$n-1'$	$n-2'$	$n-3'$	$\ldots$	$0'$	$n-1$	$n-2$	$n-3$	$\ldots$	0

Figure 1. The dihedral group D_n.

A transversal in D_n is given by the triples

$$(0,0,0), (1,1,2), \ldots, (\tfrac{n}{2}-1, \tfrac{n}{2}-1, n-2),$$
$$(\tfrac{n}{2}, \tfrac{n}{2}', 0'), (\tfrac{n}{2}+1, \tfrac{n}{2}+1', 2'), \ldots, (n-1, n-1', n-2'),$$
$$(0', n-1, 1'), (1', n-2, 3'), \ldots, (\tfrac{n}{2}-1', \tfrac{n}{2}, n-1'),$$
$$(\tfrac{n}{2}', \tfrac{n}{2}-1', 1), (\tfrac{n}{2}+1', \tfrac{n}{2}-2', 3), \ldots, (n-1', 0', n-1).$$

We next apply the following permutations to D_n to form H_n.

$$\text{Entries: } \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 & 0' & 1' & 2' & \ldots & n-1' \\ n-2 & n-3 & n-4 & \ldots & 0 & n-1 & 0' & 1' & 2' & \ldots & n-1' \end{pmatrix},$$

$$\text{columns: } \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 & 0' & 1' & 2' & \ldots & n-1' \\ n-1 & n-2 & n-3 & \ldots & 1 & 0 & 0' & 1' & 2' & \ldots & n-1' \end{pmatrix},$$

$$\text{rows: } \left(\begin{array}{cccccccc} 0 & 1 & 2 & \dots & \frac{n}{2}-2 & \frac{n}{2}-1 & \frac{n}{2} & \dots \\ n-2' & n-3' & n-4' & \dots & \frac{n}{2}' & \frac{n}{2}-2' & \frac{n}{2}-3' & \dots \end{array} \right.$$
$$\left. \begin{array}{ccccccccc} \dots & n-3 & n-2 & n-1 & 0' & 1' & 2' & \dots & n-1' \\ \dots & 0' & n-1' & \frac{n}{2}-1' & n-2 & n-3 & n-4 & \dots & n-1 \end{array} \right).$$

This gives H_n as shown in Figure 2.

	0	1	2	$\dots$	$n-1$	$0'$	$1'$	$2'$	$\dots$	$n-1'$
0	$n-1'$	$0'$	$1'$	$\dots$	$n-2'$	0	1	2	$\dots$	$n-1$
1	$n-2'$	$n-1'$	$0'$	$\dots$	$n-3'$	1	2	3	$\dots$	0
2	$n-3'$	$n-2'$	$n-1'$	$\dots$	$n-4'$	2	3	4	$\dots$	1
$\vdots$				$\vdots$					$\vdots$	
$n-1$	$0'$	$1'$	$2'$	$\dots$	$n-1'$	$n-1$	0	1	$\dots$	$n-2$
$0'$	2	3	4	$\dots$	1	$n-3'$	$n-2'$	$n-1'$	$\dots$	$n-4'$
$1'$	3	4	5	$\dots$	2	$n-4'$	$n-3'$	$n-2'$	$\dots$	$n-5'$
$2'$	4	5	6	$\dots$	3	$n-5'$	$n-4'$	$n-3'$	$\dots$	$n-6'$
$\vdots$				$\vdots$					$\vdots$	
$\frac{n}{2}-2'$	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}+2$	$\dots$	$\frac{n}{2}-1$	$\frac{n}{2}-1'$	$\frac{n}{2}'$	$\frac{n}{2}+1'$	$\dots$	$\frac{n}{2}-2'$
$\frac{n}{2}-1'$	0	1	2	$\dots$	$n-1$	$n-1'$	$0'$	$1'$	$\dots$	$n-2'$
$\frac{n}{2}'$	$\frac{n}{2}+1$	$\frac{n}{2}+2$	$\frac{n}{2}+3$	$\dots$	$\frac{n}{2}$	$\frac{n}{2}-2'$	$\frac{n}{2}-1'$	$\frac{n}{2}'$	$\dots$	$\frac{n}{2}-3'$
$\frac{n}{2}+1'$	$\frac{n}{2}+2$	$\frac{n}{2}+3$	$\frac{n}{2}+4$	$\dots$	$\frac{n}{2}+1$	$\frac{n}{2}-3'$	$\frac{n}{2}-2'$	$\frac{n}{2}-1'$	$\dots$	$\frac{n}{2}-4'$
$\frac{n}{2}+2'$	$\frac{n}{2}+3$	$\frac{n}{2}+4$	$\frac{n}{2}+5$	$\dots$	$\frac{n}{2}+2$	$\frac{n}{2}-4'$	$\frac{n}{2}-3'$	$\frac{n}{2}-2'$	$\dots$	$\frac{n}{2}-5'$
$\vdots$				$\vdots$					$\vdots$	
$n-2'$	$n-1$	0	1	$\dots$	$n-2$	$0'$	$1'$	$2'$	$\dots$	$n-1'$
$n-1'$	1	2	3	$\dots$	0	$n-2'$	$n-1'$	$0'$	$\dots$	$n-3'$

Figure 2. The Latin square H_n.

Our standard form for $C_2 \times C_n$ is shown in Figure 3.

	0	1	2	$\dots$	$n-1$	$0'$	$1'$	$2'$	$\dots$	$n-1'$
0	0	1	2	$\dots$	$n-1$	$0'$	$1'$	$2'$	$\dots$	$n-1'$
1	1	2	3	$\dots$	0	$1'$	$2'$	$3'$	$\dots$	$0'$
2	2	3	4	$\dots$	1	$2'$	$3'$	$4'$	$\dots$	$1'$
$\vdots$				$\vdots$					$\vdots$	
$n-1$	$n-1$	0	1	$\dots$	$n-2$	$n-1'$	$0'$	$1'$	$\dots$	$n-2'$
$0'$	$0'$	$1'$	$2'$	$\dots$	$n-1'$	0	1	2	$\dots$	$n-1$
$1'$	$1'$	$2'$	$3'$	$\dots$	$0'$	1	2	3	$\dots$	0
$2'$	$2'$	$3'$	$4'$	$\dots$	$1'$	2	3	4	$\dots$	1
$\vdots$				$\vdots$					$\vdots$	
$n-1'$	$n-1'$	$0'$	$1'$	$\dots$	$n-2'$	$n-1$	0	1	$\dots$	$n-2$

Figure 3. The group $C_2 \times C_n$.

Using Figures 2 and 3, we can construct the rotations at each vertex of the embedding. The rotation at a row vertex alternates column and entry vertices, so to prove that it is a single cycle of length $4n$, it suffices to list the entry vertices and to verify that these form a single cycle of length $2n$. So, consider first the rotation at the row vertex i where $0 \le i \le n-1$. The sequence(s) of entry vertices around this row vertex is (are) determined by the following permutation given in two-line form, where the top line is the i^{th} row of $C_2 \times C_n$ and the bottom line is the i^{th} row of H_n.

$$\begin{pmatrix} i & i+1 & \dots & j & \dots & i-1 & i' & i+1' & \dots & i-1' \\ n-i-1' & n-i' & \dots & n-2i-1+j' & \dots & n-i-2' & i & i+1 & \dots & i-1 \end{pmatrix}.$$

This gives $(i, -i-1', -i-1, -3i-2', -3i-2, \dots, -(2r-1)i-r, \dots)$. To see that this is a cycle of length $2n$, consider the general undashed or unprimed term (i.e. term without $'$) which is $-(2r-1)i-r$. This reduces to i modulo n if and only if $(2i+1)r = 0$ in $\mathbb{Z}_n$, and this requires that $r \equiv 0 \pmod{n}$. Hence the undashed terms form a cycle of length n and the entire permutation is a single cycle of length $2n$.

Next consider the rotation at a row vertex i' where $0 \le i \le \frac{n}{2} - 2$. The corresponding sequence(s) of entry vertices around this row vertex is (are) given by

$$\begin{pmatrix} i' & i+1' & \dots & i-1' & i & i+1 & \dots & j & \dots & i-1 \\ i+2 & i+3 & \dots & i+1 & n-i-3' & n-i-2' & \dots & n-2i-3+j' & \dots & n-i-4' \end{pmatrix}.$$

This gives $(i, -i-3', -i-1, -3i-4', -3i-2, \dots, -(2r-1)i-r, \dots)$, which again forms a single cycle of length $2n$. The same argument applies to a row vertex i' where $\frac{n}{2} \le i \le n-2$. It remains to consider rows $\frac{n}{2}-1'$ and $n-1'$. Corresponding to the former is the permutation

$$\begin{pmatrix} \frac{n}{2}-1' & \frac{n}{2}' & \dots & \frac{n}{2}-2' & \frac{n}{2}-1 & \frac{n}{2} & \dots & \frac{n}{2}-2 \\ 0 & 1 & \dots & n-1 & n-1' & 0' & \dots & n-2' \end{pmatrix}.$$

This reduces to the single cycle $(0, \frac{n}{2}', 1, \frac{n}{2}+1', 2, \dots)$. For row $n-1'$ we have the permutation

$$\begin{pmatrix} n-1' & 0' & \dots & n-2' & n-1 & 0 & \dots & n-2 \\ 1 & 2 & \dots & 0 & n-2' & n-1' & \dots & n-3' \end{pmatrix}.$$

This reduces to the single cycle $(0, -1', 1, 0', 2, \dots)$. Thus the rotation at each row vertex is a single cycle of length $4n$.

Next we consider the rotations at the column vertices. Note first that the columns of both $C_2 \times C_n$ and H_n have a cyclic pattern of order n. To be precise, for $1 \le i \le n-1$, column i (i') of both $C_2 \times C_n$ and H_n may be obtained by adding i to the entries in column 0 $(0')$. It therefore suffices to show that the rotations at column vertices 0 and $0'$ are single cycles. Again we give the sequence(s) of entry vertices around column vertices 0 and $0'$ by means of permutations in two-line form. For column vertex 0 this permutation is

$$\begin{pmatrix} 0 & 1 & \dots & n-1 & 0' & 1' & \dots & \frac{n}{2}-2' & \frac{n}{2}-1' & \frac{n}{2}' & \dots & n-2' & n-1' \\ n-1' & n-2' & \dots & 0' & 2 & 3 & \dots & \frac{n}{2} & 0 & \frac{n}{2}+1 & \dots & n-1 & 1 \end{pmatrix}.$$

For $n = 4$ the reader can easily check that this is a single cycle. In general, it gives

$$(0, -1', 1, -2', -1, 0', 2, -3', -2, 1', 3, -4', -3, 2', 4, \dots, \tfrac{n}{2}-1, \tfrac{n}{2}', \tfrac{n}{2}+1, \tfrac{n}{2}-2', \tfrac{n}{2}, \tfrac{n}{2}-1'),$$

where sufficient terms are listed for the pattern to be apparent, and this is a cycle of length $2n$. For column $0'$, the permutation is

$$\begin{pmatrix} 0' & 1' & \dots & n-1' & 0 & 1 & \dots & \frac{n}{2}-2 & \frac{n}{2}-1 & \frac{n}{2} & \dots & n-2 & n-1 \\ 0 & 1 & \dots & n-1 & n-3' & n-4' & \dots & \frac{n}{2}-1' & n-1' & \frac{n}{2}-2' & \dots & 0' & n-2' \end{pmatrix}.$$

The case $n = 4$ is easily checked and the general pattern is given by

$$(0, -3', -3, 1', 1, -4', -4, 2', \dots, \tfrac{n}{2}, \tfrac{n}{2}-2', \tfrac{n}{2}-2, \tfrac{n}{2}-1', \tfrac{n}{2}-1, -1', -1, -2', -2, 0'),$$

which is again a cycle of length $2n$. Thus the rotation at each column vertex is a single cycle of length $4n$.

Finally we consider the rotations at the entry vertices. Each such rotation alternates row and column vertices, and if the row vertices form a cycle of length $2n$ then the rotation will form a single cycle of length $4n$. We start by considering entry vertex 0 and constructing a permutation representing the sequence(s) of row vertices around this entry vertex. In two-line format this is given by taking the top line to be the list, in column order, of the row labels corresponding to the entry 0 in $C_2 \times C_n$, and the bottom line is formed similarly from H_n. The permutation is as follows.

$$\begin{pmatrix} 0 & n-1 & n-2 & \dots & \frac{n}{2}+1 & \frac{n}{2} & \frac{n}{2}-1 & \dots \\ \frac{n}{2}-1' & n-2' & n-3' & \dots & \frac{n}{2}' & \frac{n}{2}-2' & \frac{n}{2}-3' & \dots \end{pmatrix.}$$
$$\begin{matrix} \dots & 2 & 1 & 0' & n-1' & n-2' & \dots & 1' \\ \dots & 0' & n-1' & 0 & n-1 & n-2 & \dots & 1 \end{matrix} \Bigg).$$

Again $n = 4$ is easily checked, and in general this reduces to

$$(0, \tfrac{n}{2}-1', \tfrac{n}{2}-1, \tfrac{n}{2}-3', \tfrac{n}{2}-3, \dots, 3', 3, 1', 1, -1', -1, -2', -2, -3', \dots$$
$$\dots, \tfrac{n}{2}+1, \tfrac{n}{2}', \tfrac{n}{2}, \tfrac{n}{2}-2', \tfrac{n}{2}-2, \dots, 2', 2, 0'),$$

which is a cycle of length $2n$. Noting the cyclic pattern of the columns of the two Latin squares and to which we have already referred, it follows that exactly the same permutation will be found for each entry i with $0 \le i \le n-1$. Turning attention to the entry vertex $0'$ in a similar fashion, the case $n = 4$ is easily checked, and for $n \ge 8$ we obtain the permutation

$$\Bigg(\begin{matrix} 0' & n-1' & n-2' & \dots & 1' & 0 & n-1 & n-2 & n-3 & n-4 & \dots \\ n-1 & 0 & 1 & \dots & n-2 & n-2' & \frac{n}{2}-1' & n-1' & 0' & 1' & \dots \end{matrix}$$
$$\begin{matrix} \dots & \frac{n}{2}+1 & \frac{n}{2} & \frac{n}{2}-1 & \frac{n}{2}-2 & \dots & 2 & 1 \\ \dots & \frac{n}{2}-4' & \frac{n}{2}-3' & \frac{n}{2}-2' & \frac{n}{2}' & \dots & n-4' & n-3' \end{matrix} \Bigg).$$

This reduces to

$$(0, -2', 1, -3', 2, \dots, \tfrac{n}{2}-2, \tfrac{n}{2}', \tfrac{n}{2}-1, \tfrac{n}{2}-2', \tfrac{n}{2}+1, \tfrac{n}{2}-4', \tfrac{n}{2}+3, \dots$$
$$\dots, -3, 0', -1, \tfrac{n}{2}-1', \tfrac{n}{2}, \tfrac{n}{2}-3', \tfrac{n}{2}+2, \tfrac{n}{2}-5', \tfrac{n}{2}+4, \dots, -2, -1'),$$

which is a cycle of length $2n$. The same permutation will be found for each entry i' with $0 \le i \le n-1$. Thus the rotation at each entry vertex is a single cycle of length $4n$. □

We remark that for $n = 2^i$ ($i \ge 2$), Lemma 2.3 also provides a biembedding of the dihedral group D_n with a copy of $C_2 \times C_n$, and it is easily seen that this latter square also has a transversal.

Lemma 2.4 *If $t \ge 3$ is odd then $C_{2t} \bowtie H_t+$ for some H_t having a transversal.*

Proof. Note that, as groups, $C_{2t} = C_2 \times C_t$. The regular biembedding of C_t is $C_t \bowtie C_t'$, where C_t and C_t' are as given in Theorem 2.4. Since t is odd, C_t' has a transversal $\mathcal{T} = \{(i, i, 2i+1) : i \in \mathbb{Z}_t\}$. Now apply Theorem 2.2 to form the embedding $C_t(C_2) \bowtie C_t'(C_2, \mathcal{T}, C_2')$. But $C_t(C_2)$ is just a Cayley table for C_{2t}, and so the result will follow once it is shown that $C_t'(C_2, \mathcal{T}, C_2')$ has a transversal. This Latin square has the general form shown in Figure 4. It has a transversal $\{(2i, 2i+1, 4i+2), (2i+1, 2i+2, 4i+5) : i \in \mathbb{Z}_t\}$ which is shown highlighted in the Figure.

	0	1	2	3	4	5	…	$2t-2$	$2t-1$
0	3	2	4	5	6	7	…	0	1
1	2	3	5	4	7	6	…	1	0
2	4	5	7	6	8	9	…	2	3
3	5	4	6	7	9	8	…	3	2
4	6	7	8	9	11	10	…	4	5
5	7	6	9	8	10	11	…	5	4
⋮							⋮		
$2t-2$	0	1	2	3	4	5	…	$2t-1$	$2t-2$
$2t-1$	1	0	3	2	5	4	…	$2t-2$	$2t-1$

Figure 4. The Latin square $C_t'(C_2, \mathcal{T}, C_2')$. □

Proof of Theorem 2.1. Suppose that G is an Abelian group. In general, we may write G as a direct product of cyclic groups in the form

$$G = C_{2^{i_1}}^{j_1} \times C_{2^{i_2}}^{j_2} \times \cdots \times C_{2^{i_m}}^{j_m} \times C_{k_1}^{l_1} \times C_{k_2}^{l_2} \times \cdots \times C_{k_n}^{l_n},$$

where each i_s, j_s and l_s is a positive integer, and each k_s is an odd positive integer. Without loss of generality we may assume that $i_1 < i_2 < \ldots < i_m$ and $k_1 < k_2 < \ldots < k_n$. If G has no factor C_{2^i}, that is if $m = 0$, then starting with the regular biembedding of each C_{k_s} and applying Theorem 2.2 repeatedly, we have $G \bowtie H+$ for some H. In view of Lemma 2.1, the same is true if G has factors C_{2^i} for $i \ge 3$ but no factors C_2 or C_4. It remains to deal with the cases when G has factors C_2 and/or C_4.

Consider first the case when G has no factors apart from C_2 and C_4, that is $G = C_2^{j_1} \times C_4^{j_2}$. If $(j_1, j_2) = (0, 0)$ there is nothing to prove. Other cases are dealt with in Table 2, where R denotes use of a regular biembedding (Theorem 2.4), L a lemma, and T a theorem.

j_1	j_2	G	$G \bowtie H$ (?)
0	1	C_4	R.
	≥ 2	$C_4^{j_2}$	L2.2.
1	0	C_2	R.
	1	$C_2 \times C_4$	L2.3.
	2	$C_4 \times (C_2 \times C_4)$	R, L2.3, T2.2.
	≥ 3	$C_4^{j_2-1} \times (C_2 \times C_4)$	L2.2, L2.3, T2.2.
2	0	C_2^2	No biembedding (T2.3).
	1	$C_2 \times (C_2 \times C_4)$	L2.3, T2.2.
	2	$(C_2 \times C_4) \times (C_2 \times C_4)$	L2.3, T2.2.
	3	$C_4 \times (C_2 \times C_4) \times (C_2 \times C_4)$	R, L2.3, T2.2.
	≥ 4	$C_4^{j_2-2} \times (C_2 \times C_4) \times (C_2 \times C_4)$	L2.2, L2.3, T2.2.
≥ 3	0	$C_2^{j_1}$	T2.3.
	1	$C_4 \times C_2^{j_1}$	R, T2.3, T2.2.
	≥ 2	$C_4^{j_2} \times C_2^{j_1}$	L2.2, T2.3, T2.2.

Table 2. $G = C_2^{j_1} \times C_4^{j_2}$.

Next consider the case when $G = C_2^{j_1} \times C_4^{j_2} \times G^*$ where G^* is non-trivial but has no factors C_2 or C_4. We already have $G^* \bowtie H^*+$ for some H^*, so if $(j_1, j_2) \neq (2, 0)$, by using Theorem 2.2 and the appropriate case from Table 2, we have $G \bowtie H$ for some H. All that remains to consider is the case $C_2^2 \times G^*$. Since G^* is non-trivial, it has a factor C_t where $t \geq 3$ is either odd or a power of 2, not 2 or 4, so that $G^* = C_t \times \bar{G}$, where $\bar{G}$ may or may not be trivial. Since $\bar{G}$ has no factor C_2 or C_4, if $\bar{G}$ is non-trivial then $\bar{G} \bowtie \bar{H}+$ for some $\bar{H}$. In any case, we may write $G = C_2 \times (C_2 \times C_t) \times \bar{G}$. If $t = 2^i$ with $i \geq 3$, then apply Lemma 2.3 and Theorem 2.2. If t is odd, apply Lemma 2.4 and Theorem 2.2. This completes the proof of Theorem 2.1. $\square$

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