A census of the orientable biembeddings of Steiner triple systems of order 15

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Abstract

A complete census is given of the orientable biembeddings of Steiner triple systems of order 15. There are 80 Steiner triple systems of order 15 and these generate a total of 9530 orientable biembeddings.

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1 Introduction

This paper is concerned with biembeddings of Steiner triple systems. A Steiner triple system of order \( n \), \( \text{STS}(n) \), is a pair \((V, B)\), where \( V \) is a set of \( n \) points and \( B \) is a collection of triples, also called blocks, taken from \( V \) and such that every pair of distinct points from \( V \) appears in precisely one block. Such systems exist if and only if \( n \equiv 1 \) or \( 3 \mod 6 \) \([9]\). Biembeddings of such systems in orientable and nonorientable surfaces arise as follows. Consider a triangular embedding \( M \) of the complete graph \( K_n \) in the orientable surface \( S_g \), the sphere with \( g \) handles, or in the nonorientable surface \( N_\gamma \), the sphere with \( \gamma \) crosscaps. If the faces of \( M \) can be properly 2-coloured then the faces in each colour class form an \( \text{STS}(n) \).

Euler’s formula gives \( g \) (respectively \( \gamma \)) for triangular embeddings of \( K_n \) in an orientable (respectively nonorientable) surface, namely \( g = (n - 3)(n - 4)/12 \) (\( \gamma = (n - 3)(n - 4)/6 \)). Face 2-colourability requires \( n \) to be odd. It easily follows that a necessary condition for an orientable biembedding is that \( n \equiv 3 \) or \( 7 \mod 12 \), and a necessary condition for a nonorientable biembedding is \( n \equiv 1 \) or \( 3 \mod 6 \). These necessary conditions are also sufficient for the existence of such biembeddings, except in the case \( n = 7 \), where there is no biembedding of \( \text{STS}(7) \)s in the surface \( N_2 \) \([11, 12, 6]\).

Two \( \text{STS}(n) \)s, \( A \) and \( B \), are said to be biembeddable in some surface if there exists a face 2-colourable triangular embedding of \( K_n \) in that surface in which the Steiner triple systems arising from the two colour classes are isomorphic copies of \( A \) and \( B \). The issue which then naturally arises is to determine which pairs of \( \text{STS}(n) \)s are biembeddable. For \( v = 3 \) there is a trivial and unique embedding of a triangle in the sphere, and this provides a biembedding of \( \text{STS}(3) \)s. The \( \text{STS}(7) \) is unique and there is precisely one biembedding of the system, this with an isomorphic copy of itself, in the torus \( S_1 \). Here and subsequently when enumerating, we refer to the number of isomorphism classes. For the \( \text{STS}(9) \), which is also unique, there is again precisely one biembedding, again with an isomorphic copy of itself, in the nonorientable surface \( N_5 \) \([1, 5]\). There are two \( \text{STS}(13) \)s, one is cyclic and the other is not. We will refer to these here as \( C \) and \( N \) respectively. There are 615 biembeddings of \( C \) with \( C \), 8 539 biembeddings of \( C \) with \( N \), and 29 454 biembeddings of \( N \) with \( N \) \([7]\), all of these being in the nonorientable surface \( N_{15} \).

Biembeddings of \( \text{STS}(15) \)s lie in either the orientable surface \( S_{11} \) or the nonorientable surface \( N_{22} \). There are 80 isomorphism classes of \( \text{STS}(15) \)s and we follow the standard numbering of these given in \([10]\) which also gives the orders of their automorphism groups. It was shown in \([3]\) that every pair, including isomorphic pairs, has a biembedding in a nonorientable surface, although a complete enumeration of all nonorientable biembeddings of \( \text{STS}(15) \)s is probably beyond current computational capabilities.

It was proved in \([2]\) that at least one pair, namely \{\#1, \#2\} in the standard numbering, has no biembedding in an orientable surface. In \([4]\) a search was made...
to find an orientable self-embedding, that is a biembedding of two isomorphic copies of the same system, for each of the 80 STS(15)s. The biembedding was assumed to have an involutory automorphism, with a single fixed point, that reversed the colour classes. Self-embeddings of this type were found for 78 of the 80 systems. It was also shown in [4] that there is no orientable self-embedding of system #2, whether of this type or any other type. The other exception was system #79. In [8] it was shown that this latter system also has no orientable self-embedding. In the same paper we gave all orientable biembeddings in which one of the two systems is one of #1, #2, #76, #79, #80, these systems being of particular interest for reasons given in that paper. In the current paper we complete the determination of all the orientable biembeddings of the 80 Steiner triple systems of order 15.

2 Method

The biembeddings were found by one of the two computer programs used in [8]. This program takes two systems $A$ and $B$ from the listing of [10]. The representation of system $B$ is not altered, but permutations are in turn applied to the vertices of system $A$. For each permutation, two sets of triangles result, those from $A$ and those from $B$. To form a surface embedding we have, for example, the trivial test that these sets of triangles must be disjoint. During the examination of possible permutations, the automorphism groups of the two systems are exploited. If we examine all permutations $\pi$ that take a point $a$ of $A$ to a point $b$ of $B$, then we do not need to consider further any other permutation $\pi'$ that takes a point in the orbit of $a$ under $\text{Aut}(A)$ to a point in the orbit of $b$ under $\text{Aut}(B)$.

The permutations were constructed as follows. One vertex, say $b_1$, is chosen. Assume that the triples of $B$ containing $b_1$ are $b_1b_2b_3$, $b_1b_4b_5$, $\ldots$, $b_1b_{14}b_{15}$. Then in any biembedding, in the rotation around $b_1$ there are pairs $b_2b_3$, $b_4b_5$, $\ldots$, $b_{14}b_{15}$. Since these pairs may be reversed and mutually interchanged, there are $6! \cdot 2^6 = 46080$ possible rotations at $b_1$, and these are considered in turn. In any biembedding of $A$ with $B$, one vertex of $A$ is mapped to $b_1$, and some other vertex is mapped to $b_2$. But as we already have the rotation at $b_1$, the image of the third vertex in the triple from $A$ contains the inverse image of the pair $b_1b_2$ is determined. Similarly, the image of a fourth vertex determines that of a fifth vertex, and so on. In this way, only $15 \cdot 14 \cdot 12 \cdot \ldots \cdot 2 = 9676800$ permutations are constructed for each possible rotation at $b_1$. So, all together, for every pair of systems we examine $(46080) \cdot (9676800) = 445906944000$ possible permutations instead of $15! = 1307674368000$, that is approximately one-third. But in fact, by using the orbits of the automorphism groups of $A$ and $B$ as described above, a much smaller number is examined. To confirm that a biembedding is formed, we simply check that the rotation at each vertex is a single cycle of length 14. To test for orientability, we fix the orientation of the triangle $b_1b_2b_3$. This implies an
orientation for all the other triangles. Since every triangle is in three rotations, after assigning it an orientation, we check the two further occurrences.

The program described above was used in [8] and the results there were verified by a second independent program. In the current paper, as an additional check on the accuracy of the results, we computed both the biembeddings of \( A \) with \( B \) and of \( B \) with \( A \) for each pair of systems \( A \) and \( B \), and we verified that the results were identical. Furthermore the second program from the earlier paper was used to confirm all the self-embeddings.

The programs were run on a Linux cluster, with each of the 80 systems being allocated to a separate processor. Determination of all the biembeddings of each system took between two and ten weeks; the systems having the smallest automorphism groups generally taking the longest times.

3 Results

Up to isomorphism, there are 9530 orientable biembeddings of Steiner triple systems of order 15. Table 1 summarizes these biembeddings in the form of an \( 80 \times 80 \) array. The entry in the \((i, j)\) cell gives the number of orientable biembeddings of system \#\( i \) with system \#\( j \). A dot (.) indicates that no biembedding exists and the 26 letters a to z represent the numbers from 10 to 35 inclusive. The letter l (representing 21 biembeddings), which could be confused with the number 1, only appears in the \((10, 10)\) cell. The symbol + denotes an entry greater than 35. In fact + only appears on the diagonal and these entries are described in more detail in Table 2. Altogether in Table 1 there are \((80 \times 81)/2 = 3240\) cells \((i, j)\) with \(1 \leq i \leq j \leq 80\), representing distinct pairs of systems. Of these, 1150 admit no biembedding while each of the remaining 2090 admits at least one biembedding.

For reasons of space, it is not feasible to list all the biembeddings. However, these are available as a file from http://mcs.open.ac.uk/mjg47/emb15.txt. Each biembedding is described by a line of the file containing the system numbers \( i \) and \( j \), the order of the automorphism group of the biembedding and followed by a permutation \( p \). To generate the biembedding, \( p \) is applied to the standard form of system \( i \) given in [10] to produce a copy \( p(i) \). The triples of the systems \( p(i) \) and \( j \) are then taken as triangles and sewn together along common edges. As an example, the first line of the file is:

\[
1 1 10 1 8 10 3 5 2 12 7 4 6 9 11 15 13
\]

Here \( i = j = 1 \), the biembedding has 10 automorphisms, and the permutation \( p \) is taken as \( (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15) \).

\[
1 8 10 3 5 2 12 7 4 6 9 11 14 15 13
\]
Table 1. The number of biregularities of the 80 STS(15)s.
The diagonal entries of Table 1 represent 1,587 self-embeddings. There is a general tendency for the diagonal entry to be the greatest entry in the corresponding row (and column). Table 2 lists the diagonal entries from Table 1 and also gives the row (and column) sums which represent the total number of biembeddings of each system. In Table 2, the first column in each section gives the system number \( i \), the second gives the number of self-embeddings \( s \), and the third gives the total number of biembeddings \( t \).

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Table 2. Numbers of self-embeddings and biembeddings.

Overall, the value of \( t \) varies from 1 to 422. Examining the results in more detail, we observe that the systems having the largest automorphism groups have, in general, the smallest total number of biembeddings \( t \). However, if we exclude the exceptional projective system #1, the value of the product of \( t \) with the order of the automorphism group varies from 264 to 1344, a factor of approximately 5. For the 36 systems with the trivial automorphism group, this product (= \( t \), the total number of biembeddings) lies in the narrow range from 318 to 422.

Most of the 9,530 biembeddings have a trivial automorphism group and no biembedding has an automorphism group of order greater than 10. We include as automorphisms of a biembedding mappings that reverse the orientation and, for self-embeddings, those that reverse the colour classes. Table 3 analyzes the biembeddings by the order of their automorphism group \( \Gamma \).
The six biembeddings having $|\Gamma'| = 10$ comprise one self-embedding of system #1, four self-embeddings of system #76, and one self-embedding of system #80. These are all described in [8].

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**References**


