MINIMAL NON-SELFCENTRIC RADIALLY-MAXIMAL GRAPHS OF RADII 4 AND 5

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Abstract. Abstract. There is a hypothesis that a non-selfcentric radially-maximal graph of radius r has at least $3r-1$ vertices. Moreover, if it has exactly $3r-1$ vertices, then it is planar with minimum degree 1 and maximum degree 3. Using an enhanced exhaustive computer search we prove this hypothesis for $r = 4, 5$.

1. Introduction and results

Let G be a graph. Its radius and diameter are denoted by $rad(G)$ and $diam(G)$, respectively. A graph is **selfcentric** if $rad(G) = diam(G)$, otherwise it is **non-selfcentric**. We say that the graph G is **radially-maximal** if adding of any edge from its complement decreases its radius, i.e., if $rad(G \cup e) < rad(G)$ for every edge e from $\overline{G}$.

Obviously, for every r there is a radially-maximal graph of radius r, as can be shown by complete graphs (in the case $r = 1$) and even cycles (in the case $r > 1$). Both complete graphs and cycles are selfcentric graphs. One may expect that a graph is radially-maximal if it is either a very dense or a balanced (highly symmetric) one. Therefore, it is interesting that for $r \geq 3$ there are non-selfcentric radially-maximal graphs of radius r which are planar. Such graphs are neither symmetric nor dense. In fact, in [1] we have the following conjecture:

Conjecture A. *Let G be a non-selfcentric radially-maximal graph with radius $r \geq 3$ on the minimum number of vertices. Then we have*

(a) *G has exactly $3r-1$ vertices;*
(b) *G is planar;*

(c) *the maximum degree of G is 3 and the minimum degree of G is 1.*

Conjecture A was verified for the case $r = 3$ (see [1]). By an exhaustive computer search it was shown that there are just two non-selfcentric radially-maximal graphs of radius 3 on at most 8 vertices. These graphs are depicted in Figure 1. As one can see, they are planar, their minimum degree is 1, the maximum degree is 3, and each of them has exactly 8 vertices.

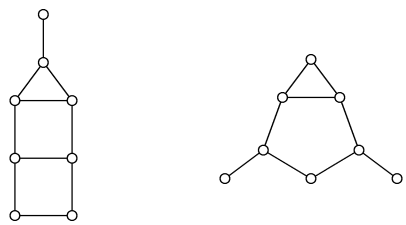


Figure 1

For higher values of r the conjecture was open. However, in [2] Haviar, Hrnčiar and Monoszová published a very nice result:

Theorem B. *Let G be a graph with $rad(G) = r$, $diam(G) \leq 2r - 2$, on at most $3r - 2$ vertices. Then G contains a geodesic cycle of length $2r$ or $2r + 1$.*

Here we recall that a cycle C in G is **geodesic**, if for any two vertices of C their distance on C equals their distance in G. In [3] we have:

Lemma C. *Let G be a radially-maximal graph of radius r and diameter d. Then $d \leq 2r - 2$.*

Theorem B and Lemma C restrict the candidates for non-selfcentric radially-maximal graphs of radius r on at most $3r-2$ vertices significantly. Using these two statements we proved the (a) part of Conjecture A for $r = 4$ (see [3]):

Theorem D. *Let G be a non-selfcentric radially-maximal graph with radius 4 on the minimum number of vertices. Then G has exactly 11 vertices.*

Consequently, using an exhaustive computer search we found that there are exactly 8 graphs of radius 4 fulfilling all the conclusions of Conjecture A. These graphs are depicted on Figure 2.

Further, generalizing the first graph of Figure 1 we obtained (see [3]):

Assertion E. *For every $r \geq 3$ there exists a non-selfcentric radially-maximal graph with radius r on $3r - 1$ vertices.*

Unfortunately, without a computer we were not able to prove more. However, as the computers are faster and faster, using an exhaustive computer search we are able to prove that the graphs depicted on Figure 2 are the only non-selfcentric radially-maximal graphs on 11 vertices. This proves Conjecture A for radius 4. Surprisingly, we are able to go even one step further. We have:

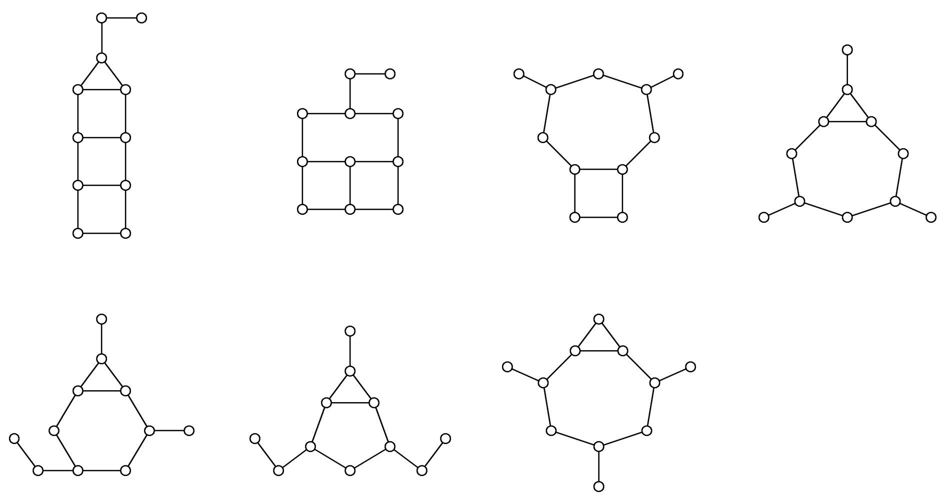
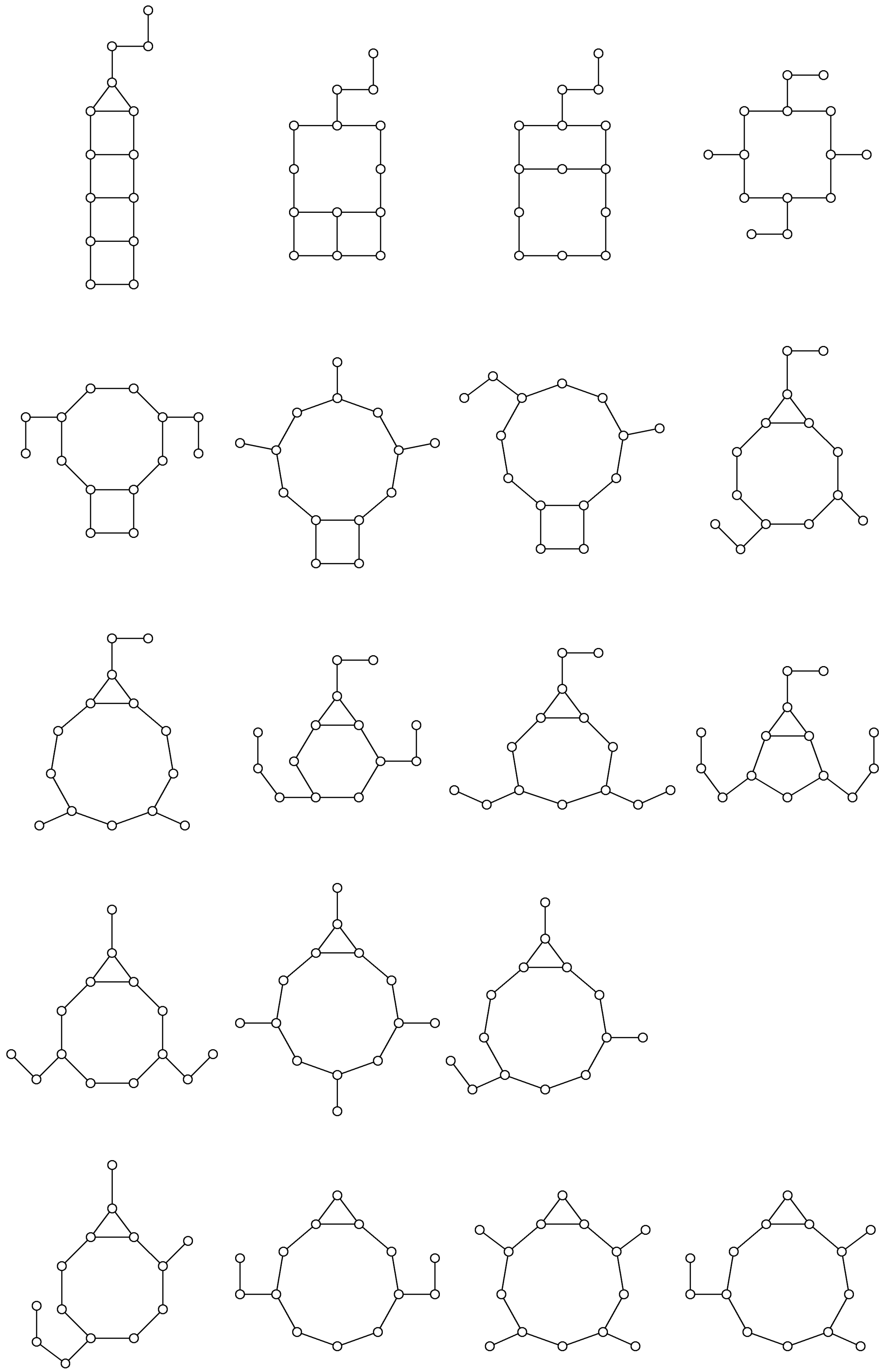


Figure 2

Theorem 1. *Conjecture A is true for the cases $r = 4$ and $r = 5$. Moreover,*

(1) *in the case $r = 4$ there are exactly 8 non-selfcentric radially-maximal graphs on 11 vertices,*
(2) *in the case $r = 5$ there are exactly 22 non-selfcentric radially-maximal graphs on 14 vertices.*

The graphs mentioned in part (2) of Theorem 1 are in Figure 3. One can observe that all these graphs are in a way similar to those on Figures 1 and 2. The only exception is the fourth graph of the first row of Figure 3, which is unicyclic. But though being non-selfcentric, the automorphism group of this graph has order 4, so that this graph is in a way a “balanced one”.



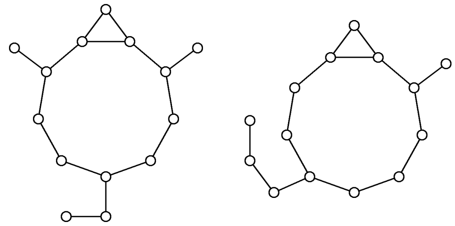


Figure 3

In the next section we describe the algorithm which found all the graphs on Figures 2 and 3.

2. Description of algorithm

In this section we describe the computer programme, which found the graphs depicted in Figures 2 and 3.

Every graph can be completed to a radially-maximal one, simply by adding edges which do not decrease the radius. This observation was used in designing the algorithm. We start with a graph, say H, and in a procedure Adding we complete its adjacency matrix, which is of size $n \times n$. More precisely, from a given pair, say (k, l) where $k < l$, we check all the entries, i.e., edges e, in the following order:

$$(v_k, v_{l+1}), (v_k, v_{l+2}), \ldots, (v_k, v_n), (v_{k+1}, v_{k+2}), (v_{k+1}, v_{k+3}), \ldots, (v_{n-1}, v_n)$$

where $v_1, v_2, \ldots, v_n$ are the vertices. Whenever the adding of e does not decrease the radius below r, we add e to the graph.

The procedure Adding finishes by examining the edge (v_{n-1}, v_n). Then we check if the graph, we just constructed, is radially-maximal. For all edges e from $\overline{G}$ the procedure Checking checks if $rad(G \cup e) < rad(G) = r$. If the check is succesfull for all edges of $\overline{G}$, the graph is radially-maximal.

In any case, we have to find another graph. A simple procedure Search finds the last edge (in the ordering mentioned above) of the adjacency matrix of $G - H$. Suppose that this last edge is (v_k, v_l). Then (v_k, v_l) is erased and we call Adding from the next entry of the adjacency matrix. If $k = n - 1$ and $l = n$, i.e., if the found edge is the last one in the ordering mentioned above, then we canot call Adding. In such a case we call Search once again. On the other hand, if Search does not find any added edge of $G - H$, the programme terminates.

This is the sketch of our computer programme, so that now we can go into details.

As we are interested in non-selfcentric graphs, we start with a path P of length $r + 1$ on vertices $v_1, v_2, \ldots, v_{r+2}$. In fact, this path is the starting graph H. And in all the programme we expect that P is a geodesic path (i.e., the distance on P equals the distance in the whole graph G). This is checked in the procedure Adding whenever we add a new edge.

In order to skip graphs which are wrong, we added to Adding yet another check. A vertex, which is eccentric to the central vertex of P, must lie outside P. We set this vertex to be v_{r+3}, and we check if the distance from $v_{\lfloor (r+3)/2 \rfloor}$ to v_{r+3} is at least r.

After some experiments we found that almost all the time is spend by a small procedure which checks the radius using the breadth-search. To shorten the running time we had to improve this procedure. This was done in two obvious steps.

First we introduced an array N in which $N(i,0)$ denotes the degree of v_i and $N(i,1), N(i,2), \ldots, N(i, N(i,0))$ denotes the $N(i,0)$ neighbours of v_i. Moreover, if $N(i,j) = k$ for $j > 0$, then in the adjacency matrix A we have $A(i,k) = j$. This enables us to address the elements in breadth-search directly.

The second step was that the bredth-search does not run from all the vertices. We skipped vertices v_j, which were already found to be at distance at least r from some v_i, where $i < j$.

Now we describe the output. Since at present there is not known a polynomial algorithm for deciding whether two graphs are isomorphic, we have to utilize the expected structure of generated graphs. We developed an invariant based on the breadth-search. Let us denote by $N_t^G(v)$ the set of vertices of G, which are at distance t from v. Further, denote by $n_t(v)$ the number of vertices in $N_t^G(v)$. To each vertex we attached the sequence $S(v) = n_1(v), n_2(v), \ldots, n_{n-1}(v)$. Now we find a vertex w which is lexicographically first acording to the sequence S. Then we list the vertices in order $w, N_1^G(w), N_2^G(w), \ldots$ so that for specific i, all the vertices of $N_i^G(w)$ are listed lexicographically with respect to S. In such a way, the 8 graphs on Figure 2 have 9 different representations and the 22 graphs on Figure 3 have 23 different representations. The graphs, which allow two different representations, are the first graph of the second row on Figure 2 and the first graph of the fifth row on Figure 3.

Using the computer programme just described, it was found that there are exactly 8 non-selfcentric radially-maximal graphs of radius 4 on 11 vertices. The running time was 11 seconds on a recent 2007 laptop in the programming language C under Linux.

For radius 5 the programme was slightly improved. Since all the vertices of P are at distance at most 3 from the vertex v_4, we included the edge (v_8, v_9) to the starting graph H. Here we recall that the distance from v_4 to v_8 has to be at least r. Finally, to get rid of redundant cases, we admit an edge (v_i, v_j) for $i \in \{1, 2, \ldots, 9\}$ and $j \in \{10, 11, \ldots, 14\}$ only if $j = 10$ or if there is $y \leq i$ such that (v_y, v_{j-1}) is an edge of our graph.

With all these improvements, the programme found that there are exactly 22 non-selfcentric radially-maximal graphs of radius 5 on 14 vertices. The running time was 52 minutes on a recent 2007 laptop in the programming language C under Linux. Of course, this was not enough to state Theorem 1. We have to prove that there are no non-selfcentric radially-maximal graphs of radius 5 on less than 14 vertices. Since a cycle is a selfcentric graph, by Theorem B and Lemma C it was enough to run our programme on 11, 12 and 13 vertices. The running time was 1, 2 and 30 seconds, respectively, and no non-selfcentric radially-maximal graph was found.

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