BIEMBEDDINGS OF SYMMETRIC CONFIGURATIONS OF TRIPLES

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Abstract

We discuss the problem of biembedding a pair of symmetric configurations of triples in a closed surface. We prove that if the system is cyclic and connected, then it is embeddable only in the torus, and the embedding is unique. We also review computer results for systems on $v \leq 10$ vertices.

1. Introduction

A *symmetric configuration* of triples is a collection of triples of a base set V which has the properties that every element or *point* of V occurs in precisely three triples or *lines* and no pair of elements occurs more than once. A symmetric configuration is *connected* if the graph of the pairs which occur is itself connected. Symmetric configurations have been the focus of study since the nineteenth century and well-known examples are the Fano plane and Pappus and Desargues configurations.

In this short note we are interested in biembeddings of symmetric configurations in orientable and non-orientable surfaces. This is equivalent to face 2-colourable triangulations of graphs in which the triangles of both colour classes form the triples of a symmetric configuration. Probably the first papers to consider this question specifically are those by Figueroa-Centeno and White, [1] and [5], where the emphasis is more on providing geometric visualisations.

From the fact that in any biembedding of symmetric configurations, each point of the base set occurs in three triangles of each of the colours, it follows that the embedded graph is regular and 6-valent. It also follows from Euler's theorem that the supporting surface is either the torus or the Klein bottle. Such embeddings have been completely classified in two papers by Negami, [3] and [4]. Hence, the problem of which pairs of symmetric configurations can be embedded is solved. From a design theoretic viewpoint however it would be nice to have an alternative proof, less topological and more combinatorial in nature, and from which the symmetric configurations can be identified more explicitly. Here we make a small contribution to this project.

A symmetric configuration on the base set V, of cardinality v, which we denote by $SC(v)$, is said to be *cyclic* if it admits a v-cycle as an automorphism. Without loss of generality we can then represent V as the set of points $\{0, 1, \ldots, v{-}1\}$ and the automorphism by mapping $i \to i{+}1 \pmod{v}$. The symmetric configuration then consists of a single orbit of a triple under the action of this automorphism. The theorem in the next section is proved by completely elementary methods.

2. Cyclic biembeddings

Theorem 1 *Let A be a connected cyclic symmetric configuration on v points. Then there is a unique biembedding of A with another symmetric configuration B. The biembedding is itself cyclic and triangulates the torus. Further, the systems A and B are isomorphic.*

Proof: The smallest possible value of v is $v = 7$. For $v = 7$ there is a unique $SC(7)$ which is the Fano plane. The unique biembedding of Fano plane with itself is well-known, and its supporting surface is torus. There are also unique cyclic $SC(v)$ for $v = 8$ and 9 and it is easily verified that these two have unique biembeddings as well.

Now suppose that $v > 9$ and let A be generated by the triple $\{0, a, a+b\}$. Without loss of generality we may assume that $a < b < v-a-b$, so that $a < \frac{v}{3}$ and $b < \frac{v}{2}$. Observe that there cannot be an equality in any of the previous inequalities, as every vertex appears in A in three triples, which must have distinct edges.

Every vertex i is in three triples in A, namely in $\{i-a-b, i-b, i\}$, $\{i-a, i, i+b\}$ and $\{i, i+a, i+a+b\}$, where all the vertices $i-a-b, i-b, i-a, i, i+a, i+b, i+a+b$ are distinct. This implies that the first neighbourhood of i contains exactly 6 vertices, which are connected by three edges $\{i-a-b, i-b\}$, $\{i-a, i+b\}$ and $\{i+a, i+a+b\}$ due to the triples containing i, and they are connected also by edges $\{i-a-b, i-a\}$, $\{i-b, i+a\}$ and $\{i+b, i+a+b\}$ due to three triples which do not contain i. (However, there may be also other edges in the first neighbourhood of i.)

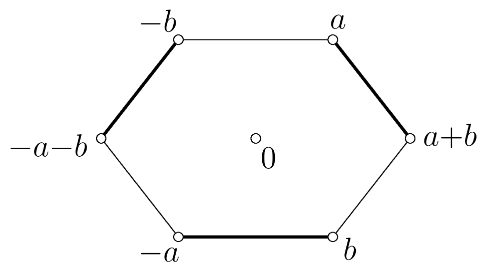

Figure 1

Consider a biembedding of A and some other system in a surface. Let us erase from the biembedding one vertex, say i. In the place of the erased vertex there appears a face which is a hexagon. Obviously, on its boundary there are edges $\{i-a-b, i-b\}$, $\{i-a, i+b\}$ and $\{i+a, i+a+b\}$. We prove that the other edges are always $\{i-a-b, i-a\}$, $\{i-b, i+a\}$ and $\{i+b, i+a+b\}$. This implies that if we fix the rotation around every vertex i to be $(i-a-b, i-b, i+a, i+a+b, i+b, i-a)$, then the triangles of the form $(j, j+a, j+a+b)$ (in the same cyclic rotation) belong to A, while those of the form $(j, j+b, j+a+b)$ belong to the other system B. Hence, the biembedding triangulates an orientable surface, i.e., torus. Moreover, the system B is isomorphic to A (it is generated by the triple $\{0, b, a+b\}$) and the biembedding is cyclic.

Suppose that erasing one vertex, say 0, gives a hexagon distinct from $(-a-b, -b, a, a+b, b, -a)$, see Figure 1 (edges in bold are the ones which form with 0 triples of A). Then in this cycle there appear at least two from the set E of 9 remaining edges, $E = \{\{-a, a\}, \{-b, b\}, \{-a-b, a+b\}, \{-b, -a\}, \{a, b\}, \{-a-b, a\}, \{-a-b, b\}, \{-b, a+b\}, \{-a, a+b\}\}$. Obviously, these edges must be already in A, so that if one of them is $\{x, y\}$, then $y-x \in \{-a-b, -b, -a, a, b, a+b\} = D$, where $|D| = 6$ as mentioned above. The case that $\{x, y\}$ lies in the hexagon and $y-x = q$ we denote by $(x, y; q) \in C$. Obviously, some of the cases cannot occur. In the following set of claims we establish the non-absurd cases.

Claim (i): If $(-a, a; q) \in C$, then the non-absurd cases are $q = -a-b$ or b.

If $q = -b$, then $a + a + b \equiv 0$, so that $a + b \equiv -a$ which contradicts $|D| = 6$.

If $q = -a$, then $a + a + a = 3a \equiv 0$, which contradicts $0 < a < \frac{v}{3}$.
If $q = a$, then $a + a - a = a \equiv 0$, which contradicts $0 < a$.
If $q = a{+}b$, then $a + a - a - b = a - b \equiv 0$, which contradicts $a < b$.

Claim (ii): If $(-b, b; q) \in C$, then the non-absurd cases are $q = -a{-}b$ or $-b$.
If $q = -a$, then $b + b + a \equiv 0$, so that $a + b \equiv -b$ which contradicts $|D| = 6$.
If $q = a$, then $b + b - a \equiv 0$, i.e., $2b \equiv a$, which contradicts $a < b < \frac{v}{2}$.
If $q = b$, then $b + b - b = b \equiv 0$, which contradicts $0 < b$.
If $q = a{+}b$, then $b + b - a - b = b - a \equiv 0$, which contradicts $a < b$.

Claim (iii): If $(-b, -a; q) \in C$ or $(a, b; q) \in C$, then the unique non-absurd case is $q = a$.
If $q = -a{-}b$, then $b - a + a + b = 2b \equiv 0$, which contradicts $b < \frac{v}{2}$.
If $q = -b$, then $b - a + b = 2b - a \equiv 0$, which contradicts $a < b < \frac{v}{2}$.
If $q = -a$, then $b - a + a = b \equiv 0$, which contradicts $0 < b$.
If $q = b$, then $b - a - b = -a \equiv 0$, which contradicts $0 < a$.
If $q = a{+}b$, then $b - a - a - b = -2a \equiv 0$, which contradicts $0 < a < \frac{v}{3}$.

Claim (iv): If $(-a{-}b, a; q) \in C$ or $(-a, a{+}b; q) \in C$, then the non-absurd cases are $q = -a{-}b$ or $-a$.
If $q = -b$, then $2a + 2b \equiv 0$, so that $a + b \equiv -a - b$ which contradicts $|D| = 6$.
If $q = a$, then $2a + b - a = a + b \equiv 0$, which contradicts $a < b < \frac{v}{2}$.
If $q = b$, then $2a + b - b = 2a \equiv 0$, which contradicts $a < \frac{v}{3}$.
If $q = a{+}b$, then $2a + b - a - b = a \equiv 0$, which contradicts $0 < a$.

Claim (v): If $(-a{-}b, b; q) \in C$ or $(-b, a{+}b; q) \in C$, then the non-absurd cases are $q = -a{-}b$ or $-b$.
If $q = -a$, then $2a + 2b \equiv 0$, so that $a + b \equiv -a - b$ which contradicts $|D| = 6$.
If $q = a$, then $a + 2b - a = 2b \equiv 0$, which contradicts $b < \frac{v}{2}$.
If $q = b$, then $a + 2b - b = a + b \equiv 0$, which contradicts $a < b < \frac{v}{2}$.
If $q = a{+}b$, then $a + 2b - a - b = b \equiv 0$, which contradicts $0 < b$.

Now we combine pairs of edges of E. We do this by distinguishing 7 cases, which exhaust all the possibilities.

Case (1): Suppose that $(-a, a; q_1), (-b, b; q_2) \in C$. By (i) and (ii), there are 4 subcases to distinguish:

(a) $q_1 = -a{-}b$ and $q_2 = -a{-}b$. Then $3a + b \equiv 0 \equiv a + 3b$, so that $2a \equiv 2b$, which contradicts $a < b < \frac{v}{2}$.

(b) $q_1 = -a{-}b$ and $q_2 = -b$. Then $3a + b \equiv 0 \equiv 3b$, so that $3a \equiv 2b$. As $a < \frac{v}{3}$ and $b < \frac{v}{2}$, we have $3a = 2b$, and hence there is k such that $a = 2k$, $b = 3k$ and $v = 9k$. Since A is a connected system, we have $k = 1$ and hence $v = 9$.

(c) $q_1 = b$ and $q_2 = -a{-}b$. Then $2a - b \equiv 0 \equiv a + 3b$. Since $a < b < \frac{v}{2}$ we have $b = 2a$. As $a + 3b < 4b < 4 \cdot \frac{v}{2} = 2v$, we have $a + 3b = v$ and consequently $v = 7a$. Since A is connected, $a = 1$ and $v = 7$.

(d) $q_1 = b$ and $q_2 = -b$. Then $2a - b \equiv 0 \equiv 3b$, so that $b = 2a$ and $v = 3b = 6a$. Since A is connected, $a = 1$ and $v = 6$.

Case (2): Suppose that $(-a, a; q_1), (-a{-}b, b; q_2) \in C$. By (i) and (v), there are 4 subcases to distinguish:

(a) $q_1 = -a-b$ and $q_2 = -a-b$. Then $3a + b \equiv 0 \equiv 2a + 3b$, so that $a \equiv 2b$, which contradicts $a < b < \frac{v}{2}$.

(b) $q_1 = -a-b$ and $q_2 = -b$. Then $3a + b \equiv 0 \equiv a + 3b$ which is already solved in (1a).

(c) $q_1 = b$ and $q_2 = -a-b$. Then $2a - b \equiv 0 \equiv 2a + 3b$, so that $b = 2a$. As $b < \frac{v}{2}$ we have $2a + 3b = 4b < 4 \cdot \frac{v}{2} = 2v$. Hence $v = 8a$. Since A is connected, $a = 1$ and $v = 8$.

(d) $q_1 = b$ and $q_2 = -b$. Then $2a - b \equiv 0 \equiv a + 3b$ which is solved in (1c).

Observe that if $(-a, a; q) \in C$, then in C we have edges $\{a+b, a\}$, $\{a, -a\}$ and $\{-a, b\}$. To complete the cycle, there is either the edge $\{b, -b\}$ or $\{b, -a-b\}$, see Figure 1. As neither of these cases is possible by (1) and (2), the edge $\{-a, a\}$ cannot be in C.

Case (3): Suppose that $(-b, b; q_1), (-a, a+b; q_2) \in C$. By (ii) and (iv), there are 4 subcases to distinguish:

(a) $q_1 = -a-b$ and $q_2 = -a-b$. Then $a + 3b \equiv 0 \equiv 3a + 2b$, so that $b = 2a$ and $v = 7a$. Since A is connected, $a = 1$ and $v = 7$.

(b) $q_1 = -a-b$ and $q_2 = -a$. Then $a + 3b \equiv 0 \equiv 3a + b$ which is solved in (1a).

(c) $q_1 = -b$ and $q_2 = -a-b$. Then $3b \equiv 0 \equiv 3a + 2b$ so that $b \equiv 3a$. As $a < \frac{v}{3}$ we have $b = 3a$, and as $3a + 2b < 3 \cdot \frac{v}{3} + 2\frac{v}{2} = 2v$ we have $v = 3a + 2b = 9a$. Since A is connected, $a = 1$ and $v = 9$.

(d) $q_1 = -b$ and $q_2 = -a$. Then $3b \equiv 0 \equiv 3a + b$ which is solved in (1b).

Observe that if $(-b, b; q) \in C$, then in C we have edges $\{-a-b, -b\}$, $\{-b, b\}$ and $\{b, -a\}$. To complete the cycle, there is either the edge $\{-a, a\}$ or $\{-a, a+b\}$, see Figure 1. As neither of these cases is possible by (1) and (3), the edge $\{-b, b\}$ cannot be in C.

Case (4): Suppose that $(-a-b, a+b; q_1), (-b, -a; q_2) \in C$. By (iii), there are 6 subcases to distinguish:

(a) $q_1 = -a-b$ and $q_2 = a$. Then $3a + 3b \equiv 0 \equiv -2a + b$, so that $b = 2a$ and $9a \equiv 0$. If $9a = v$, then the connectivity of A implies that $a = 1$ and $v = 9$. On the other hand, if $9a = 2v$ then there is k such that $a = 2k$, $v = 9k$ and $b = 4k$, and the connectivity of A implies that $k = 1$ and $v = 9$.

(b) $q_1 = -b$ and $q_2 = a$. Then $2a + 3b \equiv 0 \equiv -2a + b$ which is solved in (2c).

(c) $q_1 = -a$ and $q_2 = a$. Then $3a + 2b \equiv 0 \equiv -2a + b$, so that $b = 2a$ and $v = 7a$. Since A is connected, $a = 1$ and $v = 7$.

(d) $q_1 = a$ and $q_2 = a$. Then $a + 2b \equiv 0$, so that $a + b \equiv -b$ which contradicts $|D| = 6$.

(e) $q_1 = b$ and $q_2 = a$. Then $2a + b \equiv 0$, so that $a + b \equiv -a$ which contradicts $|D| = 6$.

(f) $q_1 = a{+}b$ and $q_2 = a$. Then $a + b \equiv 0$, which contradicts $0 < a < b < \frac{v}{2}$.

Observe that $(-a{-}b, a{+}b; q) \in C$ implies that in C there are edges $\{a, a{+}b\}$, $\{a{+}b, -a{-}b\}$ and $\{-a{-}b, -b\}$. To complete the cycle, there is either the edge $\{-b, b\}$ and consequently also $\{-a, a\}$, or there is $\{-b, -a\}$, see Figure 1. As neither of these cases is possible by (1) and (4), the edge $\{-a{-}b, a{+}b\}$ cannot be in C.

Case (5): Suppose that $(-b, -a; q_1), (-a{-}b, a; q_2) \in C$. By (iii) and (iv), there are 2 subcases to distinguish:

(a) $q_1 = a$ and $q_2 = -a{-}b$. Then $-2a + b \equiv 0 \equiv 3a + 2b$ which is solved in (4c).

(b) $q_1 = a$ and $q_2 = -a$. Then $-2a + b \equiv 0 \equiv 3a + b$, so that $b = 2a$ and $v = 5a$. Since A is connected, $a = 1$ and $v = 5$.

Observe that if $(-b, -a; q) \in C$, then in C we have edges $\{b, -a\}$, $\{-a, -b\}$ and $\{-b, -a{-}b\}$. To complete the cycle, there is either the edge $\{-a{-}b, a{+}b\}$ or there is $\{-a{-}b, a\}$, see Figure 1. As neither of these cases is possible by (4) and (5), the edge $\{-b, -a\}$ cannot be in C.

Case (6): Suppose that $(a, b; q_1), (-b, a{+}b; q_2) \in C$. By (iii) and (v), there are 2 subcases to distinguish:

(a) $q_1 = a$ and $q_2 = -a{-}b$. Then $-2a + b \equiv 0 \equiv 2a + 3b$ which is solved in (2c).

(b) $q_1 = a$ and $q_2 = -b$. Then $-2a + b \equiv 0 \equiv a + 3b$ which is solved in (1c).

Observe that if $(a, b; q) \in C$, then in C we have edges $\{-a, b\}$, $\{b, a\}$ and $\{a, a{+}b\}$. To complete the cycle, there is either the edge $\{a{+}b, -a{-}b\}$ or there is $\{a{+}b, -b\}$, see Figure 1. However, we already know that the edge $\{a{+}b, -a{-}b\}$ cannot appear in the cycle, while the other case is impossible by (6). Hence, the edge $\{a, b\}$ cannot be in C.

The edges from E, which we have not excluded yet, are $\{-a{-}b, a\}$, $\{-a{-}b, b\}$, $\{-b, a{+}b\}$ and $\{-a, a{+}b\}$. Observe that if $\{-b, a\}$ is not in C, then there must be both $\{-a{-}b, a\}$ and $\{-b, a{+}b\}$. But these two edges form a 4-cycle with $\{-a{-}b, -b\}$ and $\{a, a{+}b\}$, which is impossible. Hence, $\{-b, a\}$ is in C and consequently both $\{-a{-}b, a\}$ and $\{-b, a{+}b\}$ are missing. Thus, it remains to solve the case when C contains $\{-a{-}b, b\}$ and $\{-a, a{+}b\}$.

Case (7): Suppose that $(-a{-}b, b; q_1), (-a, a{+}b; q_2) \in C$. By (v) and (iv), there are 4 subcases to distinguish:

(a) $q_1 = -a{-}b$ and $q_2 = -a{-}b$. Then $2a + 3b \equiv 0 \equiv 3a + 2b$, so that $a \equiv b$, a contradiction.

(b) $q_1 = -a{-}b$ and $q_2 = -a$. Then $2a + 3b \equiv 0 \equiv 3a + b$ which is solved in (2a).

(c) $q_1 = -b$ and $q_2 = -a{-}b$. Then $a + 3b \equiv 0 \equiv 3a + 2b$ which is solved in (3a).

(d) $q_1 = -b$ and $q_2 = -a$. Then $a + 3b \equiv 0 \equiv 3a + b$ which is solved in (1a). $\square$

Finally, we remark that it was proved in [2] that if a 6-regular graph triangulates the torus then it is not embeddable in the Klein bottle.

3. Computer results

In the last section we summarize some computer results. There are exactly 15 $SC(v)$s for $v \leq 10$:

$Fano(7)$: (0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,0),(5,6,1),(6,0,2).
$Cyc(8)$: (0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,7),(5,6,0),(6,7,1),(7,0,2).
$Cyc(9)$: (0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,7),(5,6,8),(6,7,0),(7,8,1),(8,0,2).
$Div(9)$: (0,1,6),(1,2,4),(2,3,5),(3,4,6),(4,5,7),(5,6,8),(3,7,0),(7,8,1),(8,0,2).
$Pappus(9)$: (0,1,2),(3,4,5),(6,7,8),(0,3,7),(0,4,8),(1,3,6),(1,5,8),(2,4,6),(2,5,7).
$Cyc(10)$: (0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,7),(5,6,8),(6,7,9),(7,8,0),(8,9,1),(9,0,2).
$Div(10)$: (0,1,7),(1,2,4),(2,3,5),(3,4,6),(4,5,7),(5,6,8),(6,7,9),(3,8,0),(8,9,1),(9,0,2).
$Kant(10)$: (0,1,2),(0,3,4),(0,6,8),(1,3,5),(1,8,9),(2,4,5),(2,7,8),(3,6,9),(4,6,7),(5,7,9).
$Desar(10)$: (0,1,4),(0,2,5),(0,3,6),(1,2,7),(4,5,7),(1,3,8),(4,6,8),(2,3,9),(5,6,9),(7,8,9).
$Sts1n(10)$: (0,1,2),(1,4,6),(4,8,3),(1,7,9),(2,8,9),(0,5,6),(0,7,8),(6,7,3),(4,5,9),(2,5,3).
$Sts2n(10)$: (2,8,9),(5,7,4),(5,8,1),(0,7,8),(3,9,6),(0,9,1),(2,3,7),(3,1,4),(0,4,6),(2,5,6).
$Sts3n(10)$: (1,4,6),(0,3,4),(5,7,8),(0,5,6),(1,2,9),(6,7,9),(3,2,8),(0,8,9),(1,3,5),(4,7,2).
$Sts4n(10)$: (4,8,2),(5,7,1),(3,6,8),(5,8,0),(3,9,2),(6,7,2),(3,0,1),(6,9,1),(4,5,9),(4,7,0).
$Sts5n(10)$: (1,4,6),(2,6,0),(1,7,9),(2,8,9),(3,6,8),(5,8,0),(2,3,7),(4,5,9),(1,3,5),(4,7,0).
$Sts\_c(10)$: (1,2,5),(2,3,6),(5,6,9),(7,8,4),(0,2,7),(1,3,8),(6,8,0),(7,9,1),(9,4,3),(4,0,5).

It follows from the results of [2], [3] and [4], that each one of the above systems has at most one biembedding on either the torus or Klein bottle. By computer we found that if one of these systems is embeddable, then it embeds with itself. Such embeddings have $Fano(7)$, $Cyc(8)$, $Cyc(9)$, $Div(9)$, $Pappus(9)$ and $Cyc(10)$ with 42, 32, 18, 12, 108 and 20 automorphisms, respectively. All these embeddings are orientable (i.e., on the torus), with the unique exception of $Div(9)$, which is on the Klein bottle.

It is interesting that $Kant(10)$, $Desar(10)$, $Sts1n(10)$, $Sts2n(10)$, $Sts3n(10)$ and $Sts\_c(10)$ all contain a vertex x, such that the neighbourhood of x does not contain a cycle in which every second edge is from a triangle having x. Hence, there is no theoretical chance to biembed these systems. On the other hand, $Div(10)$, $Sts4n(10)$ and $Sts5n(10)$ do not contain such vertices, though they cannot be biembeded.

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