# LINKABILITY IN ITERATED LINE GRAPHS

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Abstract. We prove that for every graph $H$ with the minimum degree $\delta \geq 5$, the third iterated line graph $L^3(H)$ of $H$ contains $K_{\delta\lfloor\sqrt{\delta-1}\rfloor}$ as a minor. Using this fact we prove that if $G$ is a connected graph distinct from a path, then there is a number $k_G$ such that for every $i \geq k_G$ the $i$-iterated line graph of $G$ is $\frac{1}{2}\delta(L^i(G))$-linked. Since the degree of $L^i(G)$ is even, the result is best possible.



## 1. Introduction and results

Let $G$ be a graph. Its *line graph* $L(G)$ is defined as the graph whose vertices are the edges of $G$, with two vertices adjacent if and only if the corresponding edges are adjacent in $G$. Although the line graph operator is one of the most natural ones, only in recent years there is recorded a larger interest in studying iterated line graphs. *Iterated line graphs* are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

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The diameter and radius of iterated line graphs are examined in [10], and [7] is devoted to the centers of these graphs. In [3] and [2], Hartke and Higgins study the growth of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [6], and in [13] Xiong and Liu characterize the graphs whose $i$-iterated line graphs are Hamiltonian.

Note that the $i$-iterated line graph of a path on $n$ vertices is a path on $n-i$ vertices for $i < n$ and an empty graph if $i \geq n$. The iterated line graph of a cycle is isomorphic to the original cycle, and each iterated line graph of a claw $K_{1,3}$ is isomorphic to a triangle. Hence, it suffices to study connected graphs distinct from paths, cycles and the claw $K_{1,3}$. Such graphs are called *prolific*, since every two members of the sequence $\{L^i(G)\}_{i=0}^{\infty}$ are non-isomorphic.

Let $\delta(H)$ denote the minimum degree of $H$. In [3] we have:

**Theorem A.** *Let $G$ be a prolific graph. Then there is $i_G$ such that for every $i$, $i \geq i_G$, it holds that*

$$\delta(L^{i+1}(G)) = 2 \cdot \delta(L^i(G)) - 2.$$

Obviously, $\delta(L^{i_G}(G)) \geq 3$ in the above theorem. As a consequence, by the results of [6], we obtain:

**Proposition B.** *Let $G$ be a prolific graph. Then for every $i$, $i \geq i_G + 5$, the connectivity of $L^i(G)$ equals the minimum degree of $L^i(G)$.*

Here $i_G$ is the constant appearing in Theorem A.

In this paper we study the linkability of iterated line graphs. A graph with at least $2k$ vertices is said to be *$k$-linked* if for every $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ it contains $k$ vertex-disjoint paths $P_1, P_2, \ldots, P_k$, such that $P_i$ connects $s_i$ to $t_i$, $1 \leq i \leq k$.

Obviously, if a graph is $k$-linked, then it is $k$-connected. The converse is far from being true. Jung [4] and, independently, Larman and Mani [8] proved that every $2k$-connected graph that contains a subgraph isomorphic to a subdivision of $K_{3k}$ is $k$-linked. This together with a result of Mader [9] implies that for every $k$ there is an $f(k)$ such that every $f(k)$-connected graph is $k$-linked. Robertson and Seymour [11] extended the result of Jung, Larman and Mani. As a consequence of Theorem (5.4) of [11] we have:

**Proposition C.** *Every $2k$-connected graph that has a $K_{3k}$-minor is $k$-linked.*

In [1] Bollobás and Thomason proved that every $2k$-connected graph $G$ with at least $11k|V(G)|$ edges is $k$-linked. This implies that every $22k$-connected graph is $k$-linked. Recently, Thomas and Wollan [12] improved the lower bound on the number of edges in the Bollobás and Thomason result to $8k|V(G)|$. This was further improved by Kawarabayashi, Kostochka and Yu [5]. They showed that every $2k$-connected graph with average degree at least $12k$ is $k$-linked. Consequently, every $12k$-connected graph is $k$-linked.

Our main result is the following theorem:

**Theorem 1.** *Let $G$ be a prolific graph. Then there is $k_G$ such that for every $i \geq k_G$ the graph $L^i(G)$ is $\frac{1}{2}\delta(L^i(G))$-linked.*

Observe that a graph with minimum degree $\delta$ cannot be more than $\frac{1}{2}\delta$-linked if $\delta$ is even. (Consider $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ where $s_k$ is a vertex of minimum degree

$\delta = 2k-2$, and $s_1, \ldots, s_{k-1}, t_1, \ldots, t_{k-1}$ are all of the neighbours of $s_k$.) Since the minimum degree of iterated line graph $L^i(G)$ is even if $i$ is "big enough", the result of Theorem 1 is best possible.

We mention that it is an open problem to find to find "good" bounds in terms of $G$ on the numbers $i_G$ and $k_G$ in Theorem A and Theorem 1, respectively. However, if the graph $G$ is regular of degree $\delta$, then from the proof of Theorem 1 it can be deduced that $k_G \leq 11$.

In the proof of Theorem 1, which is trivially true for cycles and the claw $K_{1,3}$, we use the following statement:

**Theorem 2.** *Let $H$ be a graph with a minimum degree $\delta \geq 5$. Then $L^3(H)$ has $K_t$ as a minor, $t = \delta \cdot \lfloor\sqrt{\delta-1}\rfloor$.*

We remark that the best lower bound for the size of a complete graph in $L^3(H)$ is $4\delta - 6$. Theorem 2 shows that there exists a much larger complete graph as a minor.

## 2. Proofs

Let $G$ be a graph and let $v$ be a vertex of $L^k(G)$, $k \geq 1$. Then $v$ corresponds to an edge of $L^{k-1}(G)$, and this edge will be called 1-history of $v$. For $i \geq 2$ we define $i$-histories recursively. The $i$-history of $v$ is a subgraph of $L^{k-i}(G)$, edges of which are induced by the vertices of $L^{k-i+1}(G)$ which are in $(i-1)$-history of $v$.

Observe that 1-history is always an edge and 2-history is a path of length 2. The situation is more complicated for $i$-histories when $i \geq 3$. The only fact we can say is that $i$-history is a connected graph with at most $i$ edges, distinct from any path with less than $i$ edges, see [10]. Therefore we do not visualize the vertices of $L^3(H)$ in $H$ using their 3-histories in the proof of Theorem 2. First we use 2-histories of vertices of $L^2(H)$ and subsequently 1-histories of vertices of $L^3(H)$. In such a way, vertices of $L^3(H)$ correspond to pairs of "adjacent" 2-histories in $H$.

We prove Theorem 2 in a slightly stronger form. We prove that for an arbitrary vertex $v$ of $H$ there is a subgraph $K$ of $L^3(H)$, such that $K_t$ is a minor of $K$ and the 3-history of every vertex of $K$ contains $v$.

*Proof of Theorem 2.* Denote by $v_1, v_2, \ldots, v_\delta, \ldots$ the neighbours of $v$ in $H$.

Consider 2-histories of the vertices of $L^2(H)$ in $H$. Denote by $c_{i,i'}$ the vertex of $L^2(H)$ with 2-history $(v_i, v, v_{i'})$, and denote by $C$ the set of these vertices. Then $|C| \geq \binom{\delta}{2}$. Denote by $A_i$ those vertices of $L^2(H)$, whose 2-history have $v_i$ as a central vertex and $v$ as an endvertex. Observe that $|A_i| \geq \delta - 1$, the vertices of $A_i$ induce a complete graph in $L^2(H)$, and they are adjacent to all $c_{i,i'}$, $i' \neq i$. Moreover, the sets $A_1, A_2, \ldots, A_\delta$ are mutually disjoint.

Let $s = \lfloor\sqrt{\delta-1}\rfloor$. Equitably partition every $A_i$ into $s$ parts $A_{i,1}, A_{i,2}, \ldots, A_{i,s}$, so that $-1 \leq |A_{i,j}| - |A_{i,j'}| \leq 1$ for every $j \neq j'$. Then each $A_{i,j}$ contains at least $s$ vertices, and as $\delta \geq 5$, we have $s \geq 2$. Denote the vertices of $A_{i,j}$ by $a_{i,j,1}, a_{i,j,2}, \ldots, a_{i,j,s}, \ldots$

Now denote by $X_{i,j}$ the set of those vertices of $L^3(H)$, whose 1-histories in $L^2(H)$ contain only the vertices of $A_{i,j}$. In the following we show that there are internally vertex-disjoint paths in $L^3(H)$ connecting the sets $X_{i,j}$. Let $X_{i,j}$ and $X_{i',j'}$ be two such sets, $(i,j) \neq (i',j')$. There are two cases to distinguish:

Case 1: $i = i'$. We join the vertex of $X_{i,j}$ with 1-history $(a_{i,j,1}, a_{i,j,2})$ with the vertex of $X_{i,j'}$ with 1-history $(a_{i,j',1}, a_{i,j',2})$ by a path of length two. Its interior vertex has 1-history $(a_{i,j,1}, a_{i,j',1})$.

Case 2: $i \neq i'$. We join the vertex of $X_{i,j}$ with 1-history $(a_{i,j,1}, a_{i,j,j'})$ with a vertex of $X_{i',j'}$ with 1-history $(a_{i',j',1}, a_{i',j',j})$ by a path of length three. Its interior vertices have 1-histories $(a_{i,j,j'}, c_{i,i'})$ and $(c_{i,i'}, a_{i',j',j})$.

Obviously, the paths just constructed in $L^3(H)$ are disjoint. If we contract the vertices of $X_{i,j}$ into a single vertex $x_{i,j}$, $1 \leq i \leq \delta$ and $1 \leq j \leq s$, then the vertices $x_{i,j}$ together with the constructed paths form a subdivision of $K_{\delta \cdot s}$. Now the result is a consequence of the fact that all the vertices in $X_{i,j}$ and in the paths have $v$ in their 3-history. □

We remark that if $|A_{i,j}| = s$ in the previous proof, then the paths from $A_{i,j}$ to $A_{i',j'}$, where $i \neq i'$ and $j' = 1, 2, \ldots, s$, exhaust all the vertices with 1-histories $(a_{i,j,.}, c_{i,i'})$. This means that the choice $s = \lfloor \sqrt{\delta - 1} \rfloor$ is optimal if we restrict ourselves to the types of paths described in Cases 1 and 2.

Notice that the proof of Theorem 2 implies that, if $T$ is a tree with a central vertex $v$, such that $v$ and its neighbours have degree $\delta$ and all the remaining vertices are pendant, then $L^3(T)$ has $K_t$ as a minor, $t = \delta \cdot \lfloor \sqrt{\delta - 1} \rfloor$.

*Proof of Theorem 1.* Choose $k_G$ such that

$$k_G \geq i_G + 5 \qquad \text{and}$$
$$\left\lfloor \sqrt{\delta(L^{k_G - 3}(G)) - 1} \right\rfloor \geq 12,$$

where $i_G$ is the constant from Theorem A. Then for every $i \geq k_G$, it follows from Proposition B that $L^i(G)$ is $\delta(L^i(G))$-connected. Further, by Theorem A we have $\delta(L^i(G)) = 8\delta(L^{i-3}(G)) - 14$. Finally, by Theorem 2 $L^i(G)$ has a $K_t$-minor with

$$t = \delta(L^{i-3}(G)) \left\lfloor \sqrt{\delta(L^{i-3}(G)) - 1} \right\rfloor \geq \tfrac{1}{8}\big(\delta(L^i(G)) + 14\big) \cdot 12 > \tfrac{3}{2}\delta(L^i(G)).$$

By Proposition C this implies that $L^i(G)$ is $\frac{\delta(L^i(G))}{2}$-linked. □

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## References

[1] B. Bollobás, A. Thomason, *Highly linked graphs*, Combinatorica **16** (1996), 313-320.

[2] S.G. Hartke, A.W. Higgins, *Maximum degree growth of the iterated line graph*, Electron. J. Comb. **6**, No. 1, Research paper **R28**, 9p. (1999).

[3] S.G. Hartke, A.W. Higgins, *Minimum degree growth of the iterated line graph*, Ars Combinatoria **69** (2003), 275-283.

[4] H.A. Jung, *Verallgemeinerung des n-Fachen Zusammenhangs fuer Graphen*, Math. Annalen **187** (1970), 95-103.

[5] K.-I. Kawarabayashi, A. Kostochka, G. Yu, *On sufficient degree conditions for a graph to be k-linked*, (to appear in Combin. Probab. Comput.).

[6] M. Knor, Ľ. Niepel, *Connectivity of iterated line graphs*, Discrete Applied Math. **125** (2003), 255-266.

[7] M. Knor, Ľ. Niepel, Ľ. Šoltés, *Centers in iterated line graphs*, Acta Math. Univ. Comenian. **LXI, 2** (1992), 237-241.

[8] D.G. Larman, P. Mani, *On the existence of certain configurations within graphs and the 1-skeletons of polytopes*, Proc. London Math. Soc. **20** (1974), 144-160.

[9] W. Mader, *Homomorphieeigenschaften und Mittlere Kantendichte von Graphen*, Math. Annalen **174** (1967), 265-268.

[10] Ľ. Niepel, M. Knor, Ľ. Šoltés, *Distances in iterated line graphs*, Ars Combinatoria **43** (1996), 193-202.

[11] N. Robertson, P.D. Seymour, *Graph Minors XIII. The disjoint paths problem*, J. Comb. Theory (B) **63** (1995), 65-110.

[12] R. Thomas, P. Wollan, *An improved extremal function for graph linkages*, European J. Combin. **26** (2005), 309-324.

[13] L. Xiong, Z. Liu, *Hamiltonian iterated line graphs*, Discrete Math. **256** (2002), 407-422.