We prove that for every graph $H$ with the minimum degree $\delta \geq 5$, the third iterated line graph $L_3(H)$ of $H$ contains $K_{\delta \left \lfloor \sqrt{\delta - 1} \right \rfloor}$ as a minor. Using this fact we prove that if $G$ is a connected graph distinct from a path, then there is a number $k_G$ such that for every $i \geq k_G$ the $i$-iterated line graph of $G$ is $\frac{1}{2}\delta(L^i(G))$-linked. Since the degree of $L^i(G)$ is even, the result is best possible.

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The diameter and radius of iterated line graphs are examined in [10], and [7] is devoted to the centers of these graphs. In [3] and [2], Hartke and Higgins study the growth of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [6], and in [13] Xiong and Liu characterize the graphs whose $i$-iterated line graphs are Hamiltonian.

Note that the $i$-iterated line graph of a path on $n$ vertices is a path on $n-i$ vertices for $i < n$ and an empty graph if $i \geq n$. The iterated line graph of a cycle is isomorphic to the original cycle, and each iterated line graph of a claw $K_{1,3}$ is isomorphic to a triangle. Hence, it suffices to study connected graphs distinct from paths, cycles and the claw $K_{1,3}$. Such graphs are called prolific, since every two members of the sequence $\{L^i(G)\}_{i=0}^{\infty}$ are non-isomorphic.

Let $\delta(H)$ denote the minimum degree of $H$. In [3] we have:

**Theorem A.** Let $G$ be a prolific graph. Then there is $i_G$ such that for every $i$, $i \geq i_G$, it holds that

$$\delta(L^{i+1}(G)) = 2 \cdot \delta(L^i(G)) - 2.$$ 

Obviously, $\delta(L^{i_G}(G)) \geq 3$ in the above theorem. As a consequence, by the results of [6], we obtain:

**Proposition B.** Let $G$ be a prolific graph. Then for every $i$, $i \geq i_G + 5$, the connectivity of $L^i(G)$ equals the minimum degree of $L^i(G)$.

Here $i_G$ is the constant appearing in Theorem A.

In this paper we study the linkability of iterated line graphs. A graph with at least $2k$ vertices is said to be $k$-linked if for every $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ it contains $k$ vertex-disjoint paths $P_1, P_2, \ldots, P_k$, such that $P_i$ connects $s_i$ to $t_i$, $1 \leq i \leq k$.

Obviously, if a graph is $k$-linked, then it is $k$-connected. The converse is far from being true. Jung [4] and, independently, Larman and Mani [8] proved that every $2k$-connected graph that contains a subgraph isomorphic to a subdivision of $K_{3k}$ is $k$-linked. This together with a result of Mader [9] implies that for every $k$ there is an $f(k)$ such that every $f(k)$-connected graph is $k$-linked. Robertson and Seymour [11] extended the result of Jung, Larman and Mani. As a consequence of Theorem (5.4) of [11] we have:

**Proposition C.** Every $2k$-connected graph that has a $K_{3k}$-minor is $k$-linked.

In [1] Bollobás and Thomason proved that every $2k$-connected graph $G$ with at least $11k|V(G)|$ edges is $k$-linked. This implies that every $22k$-connected graph is $k$-linked. Recently, Thomas and Wollan [12] improved the lower bound on the number of edges in the Bollobás and Thomason result to $8k|V(G)|$. This was further improved by Kawarabayashi, Kostochka and Yu [5]. They showed that every $2k$-connected graph with average degree at least $12k$ is $k$-linked. Consequently, every $12k$-connected graph is $k$-linked.

Our main result is the following theorem:

**Theorem 1.** Let $G$ be a prolific graph. Then there is $k_G$ such that for every $i \geq k_G$ the graph $L^i(G)$ is $\frac{1}{2} \delta(L^i(G))$-linked.

Observe that a graph with minimum degree $\delta$ cannot be more than $\frac{1}{2} \delta$-linked if $\delta$ is even. (Consider $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ where $s_k$ is a vertex of minimum degree
\[ \delta = 2k - 2, \text{ and } s_1, \ldots, s_{k-1}, t_1, \ldots, t_{k-1} \text{ are all of the neighbours of } s_k. \] Since the minimum degree of iterated line graph \( L^i(G) \) is even if \( i \) is “big enough”, the result of Theorem 1 is best possible.

We mention that it is an open problem to find to find “good” bounds in terms of \( G \) on the numbers \( i_G \) and \( k_G \) in Theorem A and Theorem 1, respectively. However, if the graph \( G \) is regular of degree \( \delta \), then from the proof of Theorem 1 it can be deduced that \( k_G \leq 11 \).

In the proof of Theorem 1, which is trivially true for cycles and the claw \( K_{1,3} \), we use the following statement:

**Theorem 2.** Let \( H \) be a graph with a minimum degree \( \delta \geq 5 \). Then \( L^3(H) \) has \( K_\delta \) as a minor, \( t = \delta \cdot \lceil \sqrt{\delta - 1} \rceil \).

We remark that the best lower bound for the size of a complete graph in \( L^3(H) \) is \( 4\delta - 6 \). Theorem 2 shows that there exists a much larger complete graph as a minor.

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### 2. Proofs

Let \( G \) be a graph and let \( v \) be a vertex of \( L^k(G) \), \( k \geq 1 \). Then \( v \) corresponds to an edge of \( L^{k-1}(G) \), and this edge will be called 1-history of \( v \). For \( i \geq 2 \) we define \( i \)-histories recursively. The \( i \)-history of \( v \) is a subgraph of \( L^{k-i}(G) \), edges of which are induced by the vertices of \( L^{k-i+1}(G) \) which are in \( (i-1) \)-history of \( v \).

Observe that 1-history is always an edge and 2-history is a path of length 2. The situation is more complicated for \( i \)-histories when \( i \geq 3 \). The only fact we can say is that \( i \)-history is a connected graph with at most \( i \) edges, distinct from any path with less than \( i \) edges, see [10]. Therefore we do not visualize the vertices of \( L^3(H) \) in \( H \) using their 3-histories in the proof of Theorem 2. First we use 2-histories of vertices of \( L^2(H) \) and subsequently 1-histories of vertices of \( L^3(H) \). In such a way, vertices of \( L^3(H) \) correspond to pairs of “adjacent” 2-histories in \( H \).

We prove Theorem 2 in a slightly stronger form. We prove that for an arbitrary vertex \( v \) of \( H \) there is a subgraph \( K \) of \( L^3(H) \), such that \( K_t \) is a minor of \( K \) and the 3-history of every vertex of \( K \) contains \( v \).

**Proof of Theorem 2.** Denote by \( v_1, v_2, \ldots, v_\delta \) the neighbours of \( v \) in \( H \).

Consider 2-histories of the vertices of \( L^2(H) \) in \( H \). Denote by \( c_i, i' \) the vertex of \( L^2(H) \) with 2-history \((v_i, v, v_{i'})\), and denote by \( C \) the set of these vertices. Then \( |C| \geq \binom{\delta}{2} \). Denote by \( A_i \) those vertices of \( L^2(H) \), whose 2-history have \( v_i \) as a central vertex and \( v \) as an endvertex. Observe that \( |A_i| \geq \delta - 1 \), the vertices of \( A_i \) induce a complete graph in \( L^2(H) \), and they are adjacent to all \( c_{i,i'}, i' \neq i \).

Moreover, the sets \( A_1, A_2, \ldots, A_\delta \) are mutually disjoint.

Let \( s = \lceil \sqrt{\delta - 1} \rceil \). Equitably partition every \( A_i \) into \( s \) parts \( A_{i,1}, A_{i,2}, \ldots, A_{i,s} \), so that \(-1 \leq |A_{i,j}| - |A_{i,j'}| \leq 1 \) for every \( j \neq j' \). Then each \( A_{i,j} \) contains at least \( s \) vertices, and as \( \delta \geq 5 \), we have \( s \geq 2 \). Denote the vertices of \( A_{i,j} \) by \( a_{i,j,1}, a_{i,j,2}, \ldots, a_{i,j,s}, \ldots \).

Now denote by \( X_{i,j} \) the set of those vertices of \( L^3(H) \), whose 1-histories in \( L^2(H) \) contain only the vertices of \( A_{i,j} \). In the following we show that there are internally vertex-disjoint paths in \( L^3(H) \) connecting the sets \( X_{i,j} \). Let \( X_{i,j} \) and \( X_{i',j'} \) be two such sets, \((i, j) \neq (i', j') \). There are two cases to distinguish:

**Case 1:** \( i = i' \). We join the vertex of \( X_{i,j} \) with 1-history \((a_{i,j,1}, a_{i,j,2})\) with the vertex of \( X_{i,j'} \) with 1-history \((a_{i,j',1}, a_{i,j',2})\) by a path of length two. Its interior
vertex has 1-history \((a_{i,j,1}, a_{i,j',1})\).

Case 2: \(i \neq i'\). We join the vertex of \(X_{i,j}\) with 1-history \((a_{i,j,1}, a_{i,j,j'})\) with a vertex of \(X_{i',j'}\) with 1-history \((a_{i',j',1}, a_{i',j',j'})\) by a path of length three. Its interior vertices have 1-histories \((a_{i,j,j',c_{i,i'}}, a_{i',j',j'})\) and \((c_{i,i'}, a_{i',j',j'})\).

Obviously, the paths just constructed in \(L^3(H)\) are disjoint. If we contract the vertices of \(X_{i,j}\) into a single vertex \(x_{i,j}\), 1 \(\leq i \leq \delta\) and 1 \(\leq j \leq s\), then the vertices \(x_{i,j}\) together with the constructed paths form a subdivision of \(K_{\delta,s}\). Now the result is a consequence of the fact that all the vertices in \(X_{i,j}\) and in the paths have \(v\) in their 3-history. \(\square\)

We remark that if \(|A_{i,j}| = s\) in the previous proof, then the paths from \(A_{i,j}\) to \(A_{i',j'}\), where \(i \neq i'\) and \(j' = 1, 2, \ldots, s\), exhaust all the vertices with 1-histories \((a_{i,j,c_{i,i'}})\). This means that the choice \(s = \lfloor \sqrt{\delta - 1} \rfloor\) is optimal if we restrict ourselves to the types of paths described in Cases 1 and 2.

Notice that the proof of Theorem 2 implies that, if \(T\) is a tree with a central vertex \(v\), such that \(v\) and its neighbours have degree \(\delta\) and all the remaining vertices are pendant, then \(L^3(T)\) has \(K_t\) as a minor, \(t = \delta \cdot \lfloor \sqrt{\delta - 1} \rfloor\).

**Proof of Theorem 1.** Choose \(k_G\) such that

\[
k_G \geq i_G + 5 \quad \text{and} \quad \left\lfloor \sqrt{\delta(L^k_G-3(G))} - 1 \right\rfloor \geq 12,
\]

where \(i_G\) is the constant from Theorem A. Then for every \(i \geq k_G\), it follows from Proposition B that \(L^i(G)\) is \(\delta(L^i(G))\)-connected. Further, by Theorem A we have \(\delta(L^i(G)) = 8\delta(L^{i-3}(G)) - 14\). Finally, by Theorem 2 \(L^i(G)\) has a \(K_t\)-minor with

\[
t = \delta(L^{i-3}(G)) \left\lfloor \sqrt{\delta(L^{i-3}(G))} - 1 \right\rfloor \geq \frac{1}{8} (\delta(L^i(G)) + 14) \cdot 12 > \frac{3}{2} \delta(L^i(G)).
\]

By Proposition C this implies that \(L^i(G)\) is \(\frac{\delta(L^i(G))}{2}\)-linked. \(\square\)

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**References**


