Face two-colourable triangulations of K_{13}

M. J. Grannell, T. S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

M. Knor
Department of Mathematics
Faculty of Civil Engineering
Slovak University of Technology
Radlinského 11
813 68 Bratislava
SLOVAKIA

This is a preprint of an article accepted for publication in the Journal of Combinatorial Mathematics and Combinatorial Computing ©2003 (copyright owner as specified in the journal).

Abstract

Face two-colourable triangular embeddings of complete graphs K_n correspond to biembeddings of Steiner triple systems. Such embeddings exist only if n is congruent to 1 or 3 modulo 6. In this paper we present the number of these embeddings for $n = 13$.

1 Introduction and results

In 1968 Ringel and Youngs completed the proof of the Heawood Map Colour Theorem. An account can be found in [4]. In particular they proved that the complete graph K_n triangulates a surface if and only if $n \equiv 0$, 1, 3 or 4 (mod 6). For the embedding to be face two-colourable it is necessary for the vertex degrees to be even and, consequently, for n to be odd. Hence, face two-colourable embeddings may exist only if $n \equiv 1$ or 3 (mod 6).

Consideration of the Euler characteristic shows that such embeddings can be orientable only if $n \equiv 3$ or $7 \pmod{12}$.

In the case when $n \equiv 3 \pmod{12}$ the orientable triangulations of K_n found in [4] are indeed face two-colourable. Youngs [5] produces orientable triangulations of K_n by means of current assignments on ladder graphs. Amongst the variety of ladder graphs used in [5] it is possible to find, for each $n \equiv 7 \pmod{12}$, one which is bipartite (c.f. especially pages 39-44 of [5]), and this ensures that the corresponding triangular embedding is face two-colourable. Thus it is known that there are also face two-colourable orientable triangulations of K_n for all $n \equiv 7 \pmod{12}$.

Similar methods given by Ringel in [4] show that there is a face two-colourable triangulation of K_n in a nonorientable surface for every $n \equiv 3 \pmod{6}$ with $n \geq 9$. However, for some values of $n \equiv 1 \pmod{6}$, the situation is unclear. Although such embeddings seem easy to find for particular values of n, indeed they appear to be very much more plentiful than orientable embeddings, the general result seems elusive and some parts of the case $n \equiv 31 \pmod{36}$ are still apparently open.

There is evidence that the number of nonisomorphic face two-colourable triangulations of K_n grows rapidly with n. In [1] it is proved that the number of nonisomorphic face two-colourable triangulations of K_n in an orientable surface is at least $2^{n^2/54-O(n)}$ for $n \equiv 7$ or $19 \pmod{36}$, and is at least $2^{2n^2/81-O(n)}$ for $n \equiv 19$ or $55 \pmod{108}$. However, it seems that until the present paper, no face two-colourable embedding of K_{13} was known. In this paper we present the number of these triangulations (obtained by an exhaustive computer search) and discuss some of their features.

Our interest in face two-colourability stems from the observation that every edge of the embedded graph is part of the boundary of a face of each colour. Hence, each colour class of a face two-colourable triangulation of K_n forms a Steiner triple system of order n, STS(n). For this reason, face two-colourable embeddings of K_n correspond to biembeddings of STS(n)s. We here recall that an STS(n) may be formally defined as an ordered pair $(V, \mathcal{B})$, where V is an n-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every 2-element subset of V appears in precisely one triple. A necessary and sufficient condition for the existence of an STS(n) is that $n \equiv 1$ or $3 \pmod{6}$; such values of n are called *admissible*. We say that two STS(n)s are *biembedded* in a surface if there is a face two-colourable triangulation of K_n in which the face sets forming the two colour classes give isomorphic copies of the two systems.

There is a unique and trivial STS(3), a unique STS(7) (the Fano plane), and a unique STS(9) (the affine plane of order 3). There are two STS(13)s, one of which is cyclic (i.e. has an automorphism of order 13), and which we denote by C. This system has full automorphism group of order 39. The other STS(13) is non-cyclic, and it may be obtained from C by a so-called "Pasch switch". We denote the non-cyclic system by N; its automorphism group has order 6. When referring to the number of biembeddings, we mean the number of nonisomorphic biembeddings of the specified type. In references to the number of automorphisms of embeddings, we include automorphisms that exchange the colour classes or (in the orientable case) reverse the orientation.

The case $n = 3$ is trivial, there is a unique biembedding, this is orientable and has the automorphism group S_3 of order 6. The graph K_7 is not embeddable in the Klein bottle, see [4], and, as proved by Negami in [3], it has a unique embedding on the torus. This embedding is triangular, face two-colourable and regular, with the affine general linear group $AGL(1, 7)$ of order 42 as its automorphism group. A realization is obtained by taking one system with triples $013, 124, 235, 346, 450, 561, 602$ and the other obtained from this by applying the permutation $z \to 3z$ (arithmetic in $GF(7)$). In this realization the automorphism group is $\langle z \to az + b,\ a, b \in GF(7), a \neq 0\rangle$. The automorphisms of even order exchange the colour classes but preserve the orientation. Each colour class of the embedding forms a copy of the Fano plane.

There is a unique face two-colourable triangulation of K_9. This embedding is a vertex-transitive map and its group of automorphisms is $C_3 \times S_3$ of order 18. A realization is obtained by taking one system with triples $012, 345, 678, 036, 147, 258, 048, 156, 237, 057, 138, 246$ and the other obtained from this by applying the permutation $(0\ 1)(2\ 6)(4\ 7)(3)(5)(8)$. In this realization, the permutation just given together with $(0\ 6\ 7)(1\ 8\ 4\ 3\ 2\ 5)$ generate the automorphism group. The automorphisms of even order exchange the colour classes. Each colour class of the embedding forms a copy of the affine plane of order 3.

Regarding the STS(13)s, we may summarize our results as follows. There are 615 nonisomorphic biembeddings of C with C of which 36 have an automorphism group of order 2, and four have an automorphism group of order 3; the rest have only the trivial automorphism. There are 8 539 nonisomorphic biembeddings of C with N of which ten have an automorphism group of order 3, and the rest have only the trivial automorphism. Finally, there are 29 454 nonisomorphic biembeddings of N with N, of which 238 have an automorphism group of order 2, and the rest have only the trivial automorphism. Altogether we therefore obtain a total of 38 608 face two-colourable triangulations of K_{13}. In each case an automorphism of order 2 exchanges the colour classes and fixes exactly 3 vertices of K_{13}, and an automorphism of order 3 fixes a single vertex of K_{13}.

We remark that the embeddings given in [4] and [5] are produced by means of covering constructions, and these constructions produce large automorphism groups. It may be that this is the reason why the face two-colourable triangulations of K_{13} were not discovered earlier, although there is a large number of them.

In the next section we discuss some aspects of the embeddings of K_{13}, and we describe our computer programs. For further background and terminology regarding graph embeddings, we refer the reader to the books by Ringel [4] and by Gross and Tucker [2]. We shall denote by W (for white) and B (for black) the sets of triples forming the STS(n)s which appear as the colour classes of a face two-colourable triangulation of K_n.

2 Computational background

To obtain and verify our results we used two different computer programs, so that all the embeddings were generated in two different ways. By recording the numbers of realizations and isomorphism classes, we were also able to use the orbit-stabilizer theorem as an additional check on the computations.

In the first program we choose two STS(n)s, say W and B. The system W is fixed, and we permute the points of B using permutations p, q, etc., so that the sets of triangles (W, pB) represent an embedding. From W and pB we construct tables T_W and T_{pB}, in which the i-th row and j-th column contains the value k, for which (i, j, k) is a triple of the system. We write $k = T_W(i, j)$ or $k = T_{pB}(i, j)$, respectively. The tables T_W and T_{pB} are used for fast construction of the rotations of the embedding. If the rotations around all the vertices are cycles of length $n-1$, then (W, pB) represents a face two-colourable embedding of K_n in a surface. (We remark that in cases when one of the rotations contains a cycle of shorter length then (W, pB) represents an embedding in a pseudosurface.) Finally, we check each embedding as it arises for isomorphism with the current list of embeddings. Thus, (W, pB) is added to our list only if it is nonisomorphic to any of the embeddings constructed earlier.

The strategy just described is straightforward. However, it has to be improved in two details to get all the embeddings within a reasonable timescale.

The first improvement consists in rejecting all those permutations p, for which W and pB have a common triangle; if $T_W(i, j) = T_{pB}(i, j)$ for some i and j, then there is no need to construct the rotations, because W and pB cannot determine an embedding. We remark that in the case $n = 13$, out of $13! = 6\,227\,020\,800$ permutations p only 10.8% give a system pB which has all the triangles different from those of W. Moreover, if $T_W(i, j) = T_{pB}(i, j) = k$, then we overskip all the permutations p which do not change the triple (i, j, k). The permutations were generated lexicographically and this overskipping reduces to 34.6% the proportion of permutations which need to be considered.

The second improvement regards the isomorphism testing. Although the isomorphism problem is polynomial-time for embeddings, comparison of one embedding with 29 454 others (as would be required in the case $W = B = N$) is potentially very time-consuming. The testing can be accelerated by computing a set of invariants for each embedding. Obviously, having a subroutine that checks isomorphisms, it would be natural to count the number of automorphisms. Unfortunately, almost all the embeddings have the trivial group of automorphisms. Therefore we used different invariants.

Consider a fixed embedding, and denote by ρ_v a rotation around a vertex v. Since ρ_v is a cyclic permutation, for each two neighbours u_1 and u_3 of v there are n_1 and n_2 such that $u_3 = \rho_v^{n_1}(u_1)$ and $u_3 = \rho_v^{-n_2}(u_1)$ (where $1 \leq n_1, n_2 \leq n-2$ and $n_1 + n_2 = n-1$, the degree of v). Denote by $d(v; u_1, u_3)$ the minimum of n_1 and n_2. Now if $d(v; u_1, u_2) = 1$ and $d(v; u_2, u_3) = 1$, $u_1 \neq u_3$, then $d(u_2; u_1, u_3) = 2$. However if $d(v; u_1, u_2) = 2$ and $d(v; u_2, u_3) = 2$, $u_1 \neq u_3$, then $d(u_2; u_1, u_3)$ can be any number from 1 to $\frac{n-1}{2}$. Let I_v be the sum of $n-1$ numbers given by

$$I_v = \sum_{vu_2 \in E(G)} (d(u_2; u_1, u_3) : \text{where } d(v; u_1, u_2) = d(v; u_2, u_3) = 2 \text{ and } u_1 \neq u_3).$$

For $n = 13$, $\{I_v : v \in V(K_n)\}$ is a satisfactory set of invariants. (For instance, in the case $W = B = N$ it splits the 29 454 embeddings into 28 037 classes.)

To reconcile the number of realizations obtained by the program with the number of isomorphism classes, suppose that (W, pB) is an embedding. We determine the number of embeddings (W, qB) which are isomorphic to (W, pB).

Assume first that W is not isomorphic to B. If $m : (W, pB) \to (W, qB)$ is an isomorphism, then m is an automorphism of W and $mpB = qB$, so that $q^{-1}mp$ is an automorphism of B. Thus there are $|Aut(W)| \cdot |Aut(B)|$ possibilities for choosing the pair (m, q). However, if $|Aut(W, pB)| > 1$, some of these (m, q) pairs will provide automorphisms. Hence, the total number of embeddings (W, qB) which are isomorphic to (W, pB) is

$$\frac{|Aut(W)| \cdot |Aut(B)|}{|Aut(W, pB)|}.$$

Consider next the situation when W and B are isomorphic, say $B = rW$. Then we have also the case when m maps W to qB $(= qrW)$ and pB $(= prW)$ is mapped to W. In such a case, $mW = qrW$ and $mprW = W$. Thus, mpr and $m^{-1}qr$ are automorphisms of W, and counting the number of (m, q) pairs gives $|Aut(W)| \cdot |Aut(W)|$ possibilities. Again, some of these pairs may provide automorphisms, so that the total number of embeddings (W, qB) which are isomorphic to (W, pB) is

$$2 \cdot \frac{|Aut(W)|^2}{|Aut(W, pB)|}.$$

These calculations facilitate a partial check on our results by comparing the results of the calculations with the numbers of realizations obtained by the program and given below.

For $n = 7$ the unique STS(7) has an automorphism group of order 168 and there is a unique biembedding E with $|Aut(E)| = 42$. This gives $2 \cdot 168^2/42 = 1\,344$ realizations.

For $n = 9$ the unique STS(9) has an automorphism group of order 432 and there is a unique biembedding E with $|Aut(E)| = 18$. This gives $2 \cdot 432^2/18 = 20\,736$ realizations.

For $n = 13$ we have $|Aut(C)| = 39$ and $|Aut(N)| = 6$. There are three subcases.

(i) If $W = B = C$, the number of realizations is

$$2 \cdot 39^2 \cdot (575 + 36/2 + 4/3) = 1\,807\,962.$$

(ii) If $W = C$ and $B = N$, the number of realizations is

$$39 \cdot 6 \cdot (8\,529 + 10/3) = 1\,996\,566.$$

(iii) If $W = B = N$, the number of realizations is

$$2 \cdot 6^2 \cdot (29\,216 + 238/2) = 2\,112\,120.$$

All these numbers were confirmed by the program. It is interesting to note that in all three subcases the numbers of realizations are close to each other.

The second program was based on the observation that if (W, pB) and (W, qB) are isomorphic embeddings, then pB and qB can be identical (and not only isomorphic) systems. In fact, for every embedding (W, pB) there are $|Aut(B)|$ permutations q, such that the sets of triangles pB and qB are identical. In this second program we fix the white system W and its table T_W, and we construct the rows of T_B so that (W, B) is an embedding. This approach is a bit more tedious and it gives no information about the black system B (for example in the $n = 13$ case, whether it is cyclic or not), but it constructs only $1/39$ of embeddings if $B = C$ and $1/6$ of them if $B = N$.

We checked that the embeddings produced by this second program are the same as (i.e. isomorphic to) those produced by the first program. Unfortunately, generating the table T_B is so complicated, that the second program is only slightly faster than the first one. In fact, the total time for constructing the embeddings by the first program is less, as we can utilize the automorphism groups of C and N, rendering it unnecessary to consider all 13! permutations.

Acknowledgement. Part of this work was done while the third author was visiting the Department of Pure Mathematics of The Open University at Milton Keynes, U.K.; he thanks the Department for hospitality and financial support. He also acknowledges partial support for this research by the VEGA grant No. 1/6293/99.

References

[1] C. P. Bonnington, M. J. Grannell, T. S. Griggs and J. Širáň, Exponential families of non-isomorphic triangulations of complete graphs, *J. Comb. Theory Ser. B* **78** (2000), 169-184.

[2] J. L. Gross and T. W. Tucker, "Topological Graph Theory", John Wiley, New York, 1987.

[3] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, *Discrete Math.* **44** (1983), 161-180.

[4] G. Ringel,"Map color theorem", Springer-Verlag, New York and Berlin, 1974.

[5] J. W. T. Youngs, The mystery of the Heawood conjecture, in: "Graph Theory and its Applications" (B. Harris, Ed.), Acad. Press, 1970, 17-50.