ITERATED LINE GRAPHS

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The line-graph operator is one of the most natural operators in graph theory. It is hard to find a general monograph on graph theory in which a chapter on line graphs is missing. Therefore it is surprising that, unlike iterated line digraphs, papers devoted to iterated line graphs are very rare. The aim of this paper is to survey results on these graphs.

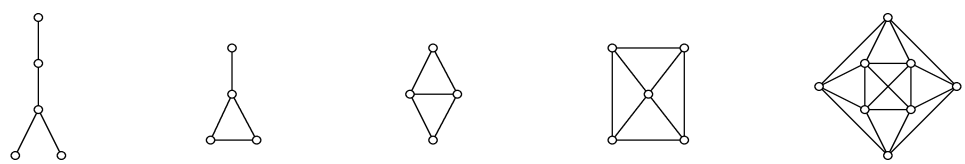
Let G be a graph. The **line graph** of G, $L(G)$, is a graph whose vertices are edges of G, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. (For other basic notions we refer to the well-known monograph of Harary [1].)

It is known that the class of line graphs is a strict subclass of the class of all graphs. Several NP-hard problems become polynomial in the class of line graphs, and for this reason line graphs are widely studied.

Iterated line graphs of G, $L^i(G)$, are defined as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0; \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In Figure 1 a sequence of iterated line graphs is depicted.



G $L(G)$ $L^2(G)$ $L^3(G)$ $L^4(G)$

Figure 1

Assertion 1 ([8]) *Let G be a connected graph.*

(i) *If G is a path of length j, then $L^i(D)$ is a path of length $j-i$ if $i \le j$, and it is an empty graph if $i > j$.*

(ii) *If G is a cycle, then each iterated line graph of G is isomorphic to the original cycle; if G is a claw $K_{1,3}$ then each iterated line graph of G is a triangle.*

(iii) *If G is a connected graph different from a path, cycle and a claw, then*

$$\lim_{i\to\infty} |V(L^i(G))| = \infty.$$

(We remark that $|V(G)|$ denotes the number of vertices of G.) By Assertion 1, it is enough to consider connected graphs different from a path, cycle and a claw. Such graphs G will be called **prolific**, since each two members of the sequence

$$G, L(G), L^2(G), \dots, L^i(G), \dots \tag{1}$$

are distinct.

Considering the sequence (1), it is natural to study the way the parameters of $L^i(G)$ depend on those of G. We begin with the minimum and the maximum degree, δ and Δ, respectively, and the number of vertices of a graph.

Assertion 2 ([8]) *For a prolific graph G and $i \geq 1$ we have*

$$2^i \cdot (\delta(G)-2)+2 \leq \delta(L^i(G)) \leq \Delta(L^i(G)) \leq 2^i \cdot (\Delta(G)-2)+2,$$

$$|V(G)| \cdot \prod_{j=0}^{i-1} \left[2^{j-1} \cdot (\delta(G)-2)+1\right] \leq |V(L^i(G))| \leq |V(G)| \cdot \prod_{j=0}^{i-1} \left[2^{j-1} \cdot (\Delta(G)-2)+1\right].$$

Moreover, equalities hold for regular prolific graphs.

By $diam(G)$ and $rad(G)$ we denote the diameter and the radius of the graph G, respectively. For line graphs we have the following theorem.

Theorem 3 ([7]) *Let G be a connected graph such that $L(G)$ is not empty. Then*

$$diam(G) - 1 \leq diam(L(G)) \leq diam(G) + 1 \quad \textit{and}$$

$$rad(G) - 1 \leq rad(L(G)) \leq rad(G) + 1\,.$$

For iterated line graphs, Theorem 3 immediately yields the bounds

$$diam(G) - i \leq diam(L^i(G)) \leq diam(G) + i.$$

Of course, this is not satisfactory. We have

Theorem 4 ([8]) *Let G be a prolific graph. Then there are i_G and t_G such that for every $i \geq i_G$ it holds that*

$$diam(L^i(G)) = i + t_G.$$

Theorem 5 ([8]) *Let G be a connected noncomplete graph with the minimum degree at least three. Then for every $i \geq 1$ we have*

$$i + diam(G) - 2 \leq diam(L^i(G)) \leq i + diam(G).$$

Unfortunately, for radius we do not have an analogue of Theorem 4.

Theorem 6 ([8]) *Let G be a prolific graph. Then there are t_G and t'_G such that for every $i \geq 0$ we have*

$$\left(i - \sqrt{2\log_2 i}\right) + t_G < rad(L^i(G)) < \left(i - \sqrt{2\log_2 i}\right) + t'_G.$$

By Theorems 4 and 6, for every prolific graph G there is a number k_G, such that if $i \geq k_G$ then $L^i(G)$ is not a selfcentric graph (i.e., the radius of $L^i(G)$ is strictly less than its diameter). Clearly, almost all graphs are prolific. Therefore the following result may be surprising.

Theorem 7 **([4])** *Let $i \geq 0$. Then for almost all graphs G we have*

$$diam(L^i(G)) = rad(L^i(G)) = i + 2.$$

We remark that Theorem 7 generalizes a well-known result which states that $diam(G) = rad(G) = 2$ for almost all graphs G, i.e., almost all graphs are selfcentric with diameter two.

Now we turn our attention to centers in iterated line graphs. It is known that each graph G can serve as a center of some graph, see [1]. We generalized this result to iterated line graphs.

Theorem 8 **([6])** *Let G be a graph and let $0 \leq j \leq 2$. If $L^j(G)$ is not empty then there is a graph H, $H \supseteq G$, such that for every i, $0 \leq i \leq j$, we have*

$$C(L^i(H)) = L^i(G).$$

Moreover, if G is triangle-free and $L^3(G)$ is not empty, then also

$$C(L^3(H)) = L^3(G).$$

Theorem 8 is best possible in a sense, since there is a graph G such that for every $i \geq 3$ and any graph H, $H \supseteq G$, we have $C(L^i(H)) \neq L^i(G)$, see [6].

We remark that Theorem 8 characterizes the centers of line graphs, since each induced subgraph of a line graph is a line graph. It means that G is a center of some line graph if and only if G is a line graph. However, the center of i-iterated line graph is not necessarily an i-iterated line graph if $i \geq 2$. Hence, the problem of characterizing the centers of i-iterated line graphs remains open for $i \geq 2$.

For vertex-connectivity κ of iterated line graphs we have

Theorem 9 **([5])** *Let G be a connected graph with the minimum degree $\delta(G) \geq 3$. Then $\kappa(L^2(G)) \geq \delta(G) - 1$.*

Theorem 10 **([5])** *Let G be a graph with $\kappa(G) \geq 4$. Then $\kappa(L^2(G)) \geq 4\delta(G) - 6$.*

Having in mind Theorem 4, the Assertion 2 is not satisfactory. Therefore, in [8] we proposed the following conjecture:

Conjecture 1 *Let G be a prolific graph. Then there is i_G such that for every $i \geq i_G$ it holds*

$$\delta(L^{i+1}(G)) = 2\delta(L^i(G)) - 2 \quad \text{and}$$
$$\Delta(L^{i+1}(G)) = 2\Delta(L^i(G)) - 2\,.$$

Recently, Hartke and Higgins proved both parts of the conjecture, see [2] for the maximum degree and [3] for the minimum one. Combining the result of [3] with Theorem 9 and 10, we have

Theorem 11 *Let G be a prolific graph. Then there is i_G such that for every $i \geq i_G$ it holds*

$$\kappa(L^i(G)) = \delta(L^i(G)).$$

By Theorem 11, the connectivity of $L^i(G)$ is the maximum possible, since it equals the minimum degree.

Finally we remark that recently Xiong and Liu have characterized Hamiltonian iterated line graphs $L^i(G)$ in terms of parameters of G, see [9].

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References

[1] Harary F., *Graph Theory*, Addison-Wesley, London, 1969.

[2] Hartke S. G., Higgins A. W., *Maximum degree growth of the iterated line graph*, Electron. J. Comb. **6**, No. 1, Research paper **R28**, 9p. (1999).

[3] Hartke S. G., Higgins A. W., *Minimum degree growth of the iterated line graph*, (submitted).

[4] Knor M., *A note on the radius in iterated line graphs*, Acta Math. Univ. Comenianae **LXIV**, **2** (1995), 255-258.

[5] Knor M., Niepel L'., *Connectivity of iterated line graphs*, (submitted).

[6] Knor M., Niepel L'., Šoltés L'., *Centers in iterated line graphs*, Acta Math. Univ. Comenianae **LXI**, **2** (1992), 237-241.

[7] Knor M., Niepel L'., Šoltés L'., *Centers in line graphs*, Math. Slovaca **43** (1993), 11-20.

[8] Niepel L'., Knor M., Šoltés L'., *Distances in iterated line graphs*, Ars Combinatoria **43** (1996), 193-202.

[9] Xiong L., Liu Z., *Hamiltonian iterated line graphs*, (submitted).

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