CENTERS IN PATH GRAPHS

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Abstract. If G is a graph, then its path graph, $P_k(G)$, has vertex set identical with the set of paths of length k in G, with two vertices adjacent in $P_k(G)$ if and only if the corresponding paths are "consecutive" in G. We prove that every path graph can serve as a center of some path graph. Moreover, we show that the class of centers of path graphs is strictly larger than the class of path graphs.

Introduction and results

Let G be a graph, $k \geq 1$, and let $\mathcal{P}_k$ be the set of all subgraphs of G which form a path of length k (i.e., with $k{+}1$ vertices). The **path graph** $P_k(G)$ of G has vertex set $\mathcal{P}_k$. Let $A, B \in \mathcal{P}_k$. The vertices of $P_k(G)$ that correspond to A and B are joined by an edge in $P_k(G)$ if and only if the edges of $A \cap B$ form a path on k vertices and $A \cup B$ is either a path of length $k{+}1$ or a cycle of length $k{+}1$.

Path graphs were investigated by Broersma and Hoede in [2], as a natural generalization of line graphs (observe that $P_1(G)$ is a line graph of G). In [5] the connectivity of path graphs is studied, and Belan and Jurica [1] bounded the diameter of path graphs. The study of path graphs has concentrated mostly on P_2-path graphs. In [2] and [10] P_2-path graphs are characterized, Yu in [12] studied the traversability of P_2-path graphs, and [6] is devoted to diameter in iterated P_2-path graphs.

By $d_G(u, v)$ we denote the distance between the vertices u and v in G. The eccentricity, $e_G(u)$, of the vertex u is the maximum $d_G(u, v)$ taken over all vertices v of G. The **radius** of a graph G is the minimum eccentricity of a vertex in G,

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and the **center**, $C(G)$, is the subgraph of G induced by vertices whose eccentricity equals the radius. It is known that each graph G can be the center of some graph H, where $|V(H)| \leq |V(G)|+4$ (see [3, p.41]). Centers of special graphs are studied in several papers. Clearly, the center of a tree consists of either a single vertex or a pair of adjacent vertices. All seven central subgraphs admissible in maximal outerplanar graphs were listed by Proskurowski [11]. Laskar and Shier [9] studied centers in chordal graphs. In [8] it is shown that every line graph can be a center of a line graph. This characterizes the centers in line graphs, as every induced subgraph of a line graph is a line graph. A survey on centers can be found in [3].

In this paper we study centers in path graphs. We prove:

Theorem 1. *Let $k \geq 1$ and let G be a graph, such that $P_k(G)$ contains at least one vertex. Then there is a supergraph H such that $C(H) = G$ and $C(P_k(H)) = P_k(G)$.*

We remark that there is a correspondence between the vertices of $P_k(G)$ and those of k-iterated line graph of G, see [4]. This correspondence implies that $P_k(G)$ is a subgraph of $L^k(G)$ (i.e., of k-iterated line graph of G). Although $P_k(G)$ is a center of $P_k(H)$ for some supergraph H of G for arbitrary k and G (such that $P_k(G)$ is not empty), by Theorem 1, there are graphs G such that $L^k(G)$ is not a center of $L^k(H)$ for any supergraph H of G and $k \geq 3$, see [7, Theorem 4].

By Theorem 1, each P_k-path graph can be a center of some P_k-path graph. However, this result does not characterize the centers of P_k-path graphs if $k \geq 2$, since not every induced subgraph of P_k-path graph is a P_k-path graph. For every $k \geq 2$ we find graphs G^k and H^k, such that G^k is the center of $P_k(H^k)$, but G^k is not a P_k-path graph. Thus, we prove:

Theorem 2. *If $k \geq 2$, then the class of centers of P_k-path graphs is strictly larger than the class of P_k-path graphs.*

We remark that it is not trivial to determine whether a given graph is a P_k-path graph at present, as only P_2-path graphs have been characterized so far.

One can ask whether every induced subgraph of a P_k-path graph can be a center of P_k-path graph. At present we do not know an answer in general. However, for $k = 2$ we have a graph, that is an induced subgraph of P_2-path graph, but cannot serve as a center of P_2-path graph. We conclude this section with two open problems:

Problem 1. *Does there exist for every $k \geq 2$ a graph, say F^k, such that F^k is an induced subgraph of a P_k-path graph, but F^k cannot be a center of P_k-path graph?*

Problem 2. *Characterize the centers of P_k-path graphs if $k \geq 2$.*

Proofs

The vertices of path graph are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G. For easier handling of paths of length k in G (i.e., the vertices of $P_k(G)$) we adopt the following convention. We denote the vertices of $P_k(G)$ (as well as the vertices of G) by small letters a, b, ... , while the corresponding paths of length k in G will be denoted by capital letters A, B, ... It means that if A is a path of length k in G and a is a vertex in $P_k(G)$, then a must be the vertex corresponding to the path A.

Let G be a graph with n vertices, and let $s \geq 1$ and $t \geq 3$ be two integer parameters. We construct a supergraph $H_{s,t}(G)$ of G in the following way.

For every vertex v of G we add a subgraph with $2[(s-1)n+2+t]$ new vertices and $2[s{\cdot}n+1+t]$ new edges. Two of the added vertices we denote by x_v^* and y_v^*. The vertices x_v^* and y_v^* are joined to every vertex of $G-\{v\}$ by a path of length s, and they are joined to v by a path of length $s{+}1$. Moreover, one extra-path of length t is glued by one endvertex to x_v^* (the other endvertex is denoted by x_v^0), and one extra-path of length t is glued by one endvertex to y_v^* (the other endvertex is denoted by y_v^0), see Figure 1 for the case $s=2$ and $t=4$. Moreover, the unique vertex at distance i from x_v^0 (from y_v^0) we denote by x_v^i (by y_v^i), $i<t$. The resulting graph is denoted by $H_{s,t}(G)$.

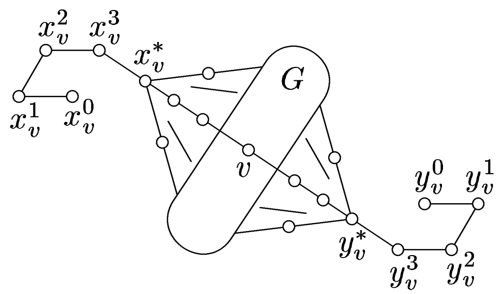

Figure 1

Lemma 3. *Let G be a graph, $k\ge 1$, and let $P_k(G)$ be a graph with at least one vertex. Further, let $s\ge \frac{k+1}{2}$ and $t\ge s+2k$. Then $C(P_k(H_{s,t}(G)))=P_k(G)$, and the radius of $P_k(H_{s,t}(G))$ equals $s+t$.*

Proof. We show that every vertex of $P_k(G)$ has eccentricity $s+t$ in $P_k(H_{s,t}(G))$, while the remaining vertices have eccentricity at least $s+t+1$. Denote by H the graph $H_{s,t}(G)$. If $v\in V(G)$, then let B_v and C_v denote the following paths of length k respectively, $(x_v^0,x_v^1,\dots,x_v^k)$ and $(y_v^0,y_v^1,\dots,y_v^k)$.

Let a be a vertex in $P_k(G)$. Then $d_{P_k(H)}(a,b_v)=s{+}t$ for arbitrary vertex v of G, as at least one endvertex of A is different from v. Hence, $e_{P_k(H)}(a)\ge s{+}t$. Now assume that A' is a path of length k in H. Let v be a vertex of G, such that the distance from one endvertex of A' to x_v^0, or to y_v^0, is the shortest possible. Assume that the shortest distance is realized by x_v^0. Then $d_{P_k(H)}(a',b_v)\le s{+}t$, since the endvertex of A' at the shortest distance from x_v^0 cannot be v. Denote $B_v^*=(x_v^*,x_v^{t-1},x_v^{t-2},\dots,x_v^{t-k})$. Then $d_{P_k(H)}(a,b_v^*)=s{+}k$. If the vertices of A' form a subset of $\{x_v^*,x_v^{t-1},x_v^{t-2}\dots,x_v^0\}$, then $d_{P_k(H)}(b_v^*,a')\le t{-}k$, so that

$$d_{P_k(H)}(a,a')\le d_{P_k(H)}(a,b_v^*)+d_{P_k(H)}(b_v^*,a')\le(s+k)+(t-k)=s+t.$$

However, if the vertices of A' do not form a subset of $\{x_v^*,x_v^{t-1},x_v^{t-2}\dots,x_v^0\}$, then $d_{P_k(H)}(b_v^*,a')\le s{+}k$ (recall that $s\ge\frac{k+1}{2}$), and hence,

$$d_{P_k(H)}(a,a')\le d_{P_k(H)}(a,b_v^*)+d_{P_k(H)}(b_v^*,a')\le(s+k)+(s+k)\le s+t,$$

as $s+2k\le t$. Thus, $e_{P_k(H)}(a)=s+t$.

Now suppose that a is a vertex of $V(P_k(H)){-}V(P_k(G))$. Then A contains an edge in $E(H){-}E(G)$, and hence, at least one endvertex of A is outside G, since $2s\ge k{+}1$. Assume that one endvertex of A (lying outside G) is in the branch containing x_v^0. Let v' be a vertex in G, $v'\neq v$. If $d_{P_k(H)}(a,c_v)\le s{+}t$, $d_{P_k(H)}(a,b_{v'})\le s{+}t$ and $d_{P_k(H)}(a,c_{v'})\le s{+}t$, then the other endvertex of A, say

u, is in G. Since $d_{P_k(H)}(a, c_u) = s{+}t{+}1$, we have $e_{P_k(H)}(a) \geq s{+}t{+}1$, and hence $C(P_k(H)) = P_k(G)$. $\square$

Proof of Theorem 1. Let $s \geq \frac{k+1}{2}$, $t \geq s{+}2k$, and let H be the graph $H_{s,t}(G)$. Since $C(P_k(H)) = P_k(G)$ by Lemma 3, it remains to show $C(H) = G$.

Suppose that $v \in V(G)$. Then $d_H(v, x_v^0) = d_H(v, y_v^0) = s + t + 1$. Moreover, $d_H(v, z) \leq s + t$ if $z \in V(H) - \{x_v^0, y_v^0\}$, as $2s + 1 \leq s + t$. Thus, $e_H(v) = s + t + 1$.

Now suppose that $v \in V(H) - V(G)$, and assume that v is in the branch containing x_u^0. If $d_H(v, y_u^0) \leq s + t + 1$, then v is adjacent to a vertex u' of G, $u \neq u'$. But then $d_H(v, x_{u'}^0) > s + t + 1$, and hence $C(H) = G$. $\square$

For $k \geq 2$ we define

$$k^o = \begin{cases} k+1 & \text{if } k \text{ is even;} \\ k+2 & \text{if } k \text{ is odd.} \end{cases}$$

Hence, k^o is an odd number.

Next lemma shows that all cycles of length k^o in $P_k(G)$ are the images (i.e., the path graphs) of cycles of length k^o in G.

Lemma 4. *Let $k \geq 2$ and let $\mathcal{C}$ be a cycle of length k^o in $P_k(G)$. Then there is a cycle $\mathcal{D}$ in G such that $P_k(\mathcal{D}) = \mathcal{C}$.*

Proof. Let $\mathcal{C} = (a_1, a_2, \ldots, a_{k^o})$. Assume that $A_1, A_2, \ldots, A_{k^o}$ are ordered so that for each i, $1 \leq i < k^0$, we have either $A_i(j) = A_{i+1}(j{+}1)$, $0 \leq j < k$, or $A_i(j) = A_{i+1}(j{-}1)$, $0 < j \leq k$ (by $A_i(j)$ we denote the j-th vertex of the path A_i, i.e., $A_i = (A_i(0), A_i(1), \ldots, A_i(k))$). Roughly speaking, the ordering of $A_1, A_2, \ldots, A_{k^o}$ has the property that a vertex common to two consecutive paths is given indices of different parity by the two paths.

First suppose that all $A_1, A_2, \ldots, A_{k^o}$ contain an edge, say e, in common. Let $e = (A_1(i_1), A_1(i_1{+}1))$ and let $A_1(i_1) = A_2(i_2) = \cdots = A_{k^o}(i_{k^o})$. Since all A_1, $A_2, \ldots, A_{k^o}$ contain the edge e, we have $A_1(i_1{+}1) = A_2(i_2{+}1) = \cdots = A_{k^o}(i_{k^o}{+}1)$. As $a_1, a_2, \ldots, a_{k^o}$ determine a walk in $P_k(G)$, all $i_1, i_3, i_5, \ldots, i_{k^o}$ have the same parity. Finally, since $A_1(i_1) = A_{k^o}(i_{k^o})$ and $A_1(i_1{+}1) = A_{k^o}(i_{k^o}{+}1)$, a_1 and a_{k^o} cannot be adjacent vertices in $P_k(G)$.

Now suppose that there is no edge common to all $A_1, A_2, \ldots, A_{k^o}$. We say that A_i is a *turning path* in the cycle $a_1, a_2, \ldots, a_{k^o}$, $1 < i < k^o$, if $A_{i-1}(j{-}1) = A_i(j) = A_{i+1}(j{-}1)$, $0 < j \leq k$, or $A_{i-1}(j{+}1) = A_i(j) = A_{i+1}(j{+}1)$, $0 \leq j < k$. We prove that there is no turning path in $a_1, a_2, \ldots, a_{k^o}$. On the contrary, suppose that A_i is a turning path, $0 < i < k^o$. Clearly, A_j and A_{j+1} have exactly $k{-}1$ edges in common, $1 \leq j < k^o$, and hence, $A_1, A_2, \ldots, A_j$ have at least $k - (j{-}1)$ edges in common, $1 \leq j \leq k^o$. However, since A_i is a turning path, A_{i-1}, A_i and A_{i+1} have exactly $k{-}1$ edges in common, too. Hence, $A_1, A_2, \ldots, A_{k^o}$ have an edge in common if k is even, a contradiction. Thus, suppose that k is odd. As shown above, A_i is the unique turning path in $a_1, a_2, \ldots, a_{k^o}$. If $3 \leq i \leq k^o{-}2$, then $A_{i-2}, A_{i-1}, A_i, A_{i+1}, A_{i+2}$ have at least $k{-}2$ edges in common, so that $A_1, A_2, \ldots, A_{k^o}$ have again an edge in common, a contradiction. Thus, suppose that $i = 2$ (the case $i = k^o{-}1$ can be solved analogously). Denote $b_1 = a_{k^o}$, $b_2 = a_1$, $b_3 = a_2$, $\ldots$, $b_{k^o} = a_{k^o-1}$. (To be precise, assume that $B_1, B_2, \ldots, B_{k^o}$ are ordered so that for each i', $1 \leq i' < k^0$, we have either $B_{i'}(j) = B_{i'+1}(j{+}1)$, $0 \leq j < k$, or $B_{i'}(j) = B_{i'+1}(j{-}1)$, $0 < j \leq k$.) Then B_3 is the unique turning path in $b_1, b_2, \ldots, b_{k^o}$. Since $k^o \geq 5$, we have

$3 \le k^o-2$. As shown above, $B_1, B_2, \ldots, B_{k^o}$ have an edge in common, and hence, also $A_1, A_2, \ldots, A_{k^o}$ have an edge in common, a contradiction.

Assume that $A_1(j) = A_2(j-1)$, $0 < j \le k$. Since there is no turning path in $a_1, a_2, \ldots, a_{k^o}$, we have $A_1(k) = A_{k+1}(0)$, and $e = (A_1(k-1), A_1(k))$ is not an edge of A_{k+1}. Since $A_{k+1}(0) = A_1(k)$, we have $A_{k^o}(j) = A_1(j-1)$, $0 < j \le k$. All $A_1(k), A_2(k), \ldots, A_{k+1}(k)$ are mutually distinct, as they are vertices of A_{k+1}. However, $A_{k^o-k}(k), A_{k^o-k+1}(k), \ldots, A_{k^o}(k)$ are mutually distinct too, as they are vertices of A_{k^o}. Since $A_1(k) \ne A_{k^o}(k)$, $\mathcal{D} = (A_1(k), A_2(k), \ldots, A_{k^o}(k))$ is a cycle of length k^o in G, and $\mathcal{C} = P_k(\mathcal{D})$. $\square$

Let G^k be a unicyclic graph on k^o+1 vertices, consisting of a cycle of length k^o and a pendant vertex glued by an edge to a vertex of the cycle, $k \ge 2$. Clearly, $P_k(G^k)$ is a graph on k^o+2 vertices, consisting of a cycle of length k^o and two edges, each glued by one endvertex to a vertex of the cycle.

Lemma 5. *If $k \ge 2$ then there is no graph G such that $P_k(G) = G^k$.*

Proof. Suppose that there is a graph G such that $P_k(G) = G^k$. Let $\mathcal{C}$ be the cycle of length k^o in G^k. By Lemma 4, G contains a cycle, say $\mathcal{D}$, of length k^o such that $P_k(\mathcal{D}) = \mathcal{C}$.

By the definition of path graphs, if a and b are adjacent vertices in $P_k(G)$, then A and B share a path of length $k-1$. Thus, at most one edge of B is not in A. The G^k contains a vertex outside $\mathcal{C}$, that is adjacent to a vertex of $\mathcal{C}$. Hence, G contains an edge, say e, glued by one endvertex to a vertex of $\mathcal{D}$ (the other endvertex of e is either on $\mathcal{D}$, or outside $\mathcal{D}$). Let G' be the graph consisting of $\mathcal{D}$ and e. Then G' contains a path A' of length k, such that a' lies outside $\mathcal{C}$. However, G' contains also a path B' of length k, such that $b' \ne a'$ and b' lies outside $\mathcal{C}$, because of the symmetry of G'. Thus, $P_k(G')$ contains more vertices than G^k, and hence, also $P_k(G)$ contains more vertices than G^k, a contradiction. $\square$

Lemma 6. *If $k \ge 2$, then there is a graph H^k such that $C(P_k(H^k)) = G^k$.*

Proof. We construct a supergraph H^k of G^k, and then we show that the center of $P_k(H^k)$ is isomorphic to G^k.

Let w be the vertex of degree one in G^k, and let z be an endvertex of a path of length k beginning in w (note that there are exactly two vertices with this property). Let $s \ge \frac{k+1}{2}$, $t \ge s+2k$, and let $H_0 = H_{s,t}(G^k)$. To H_0 we add $(s-1)n + 3 + t$ new vertices and $s{\cdot}n + 2 + t$ new edges (n is the number of vertices of G^k). One of the added vertices, say x^*, is joined to every vertex of $G^k - \{w, z\}$ by a path of length s, and x^* is joined to w and z by paths of length $s+1$. Moreover, one extra-path of length t is glued by one endvertex to x^* (the other endvertex is denoted by x^0). Let H^k be the resulting graph.

Denote by x^i the unique vertex at distance i from x^0, $i < t$. Let $B = (x^0, x^1, \ldots, x^k)$, and let A' be the path of length k in G^k with endvertices w and z (note that there is a unique path A' with this property). By Lemma 3, $C(P_k(H_0)) = P_k(G^k)$ and the radius of $P_k(H_0)$ equals $s+t$. Let a be a vertex in $P_k(G^k)$. If $a \ne a'$, then analogously as in Lemma 3 $e_{P_k(H)}(a) = s+t$ can be proved, while $d_{P_k(H)}(a', b) = s+t+1$. Moreover, analogously as in the proof of Lemma 3, one can show that the eccentricity of every vertex of $P_k(H^k) - P_k(H_0)$ exceeds $s+t$. Thus, the center of $P_k(H^k)$ is $P_k(G^k) - a'$, and hence, $C(P_k(H))$ is a graph isomorphic to G^k. $\square$

By Lemma 5, G^k is not a P_k-path graph, and hence, Theorem 2 is a corollary of Lemma 6 and Lemma 5.

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