ON RAMSEY-TYPE GAMES FOR GRAPHS

Martin KNOR

Slovak Technical University, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia

Abstract. By a Ramsey-type game is meant a game in which two players (the constructor and the destroyer) alternately pick previously unpicked edges of the complete graph on n vertices, and the constructor wins if and only if he has selected all edges of a prescribed k-vertex graph G. We prove that the constructor wins if G is an n-vertex path ($n \geq 5$) or a cycle ($n \geq 15$), or if G is an n-vertex tree having some special properties.

This is a preprint of an article accepted for publication in Australasian Journal of Combinatorics ©1996 (copyright owner as specified in the journal).

1. Introduction

The Ramsey game on pairs is a 2-player game where the players alternately pick previously unpicked edges of the complete graph on n vertices, and the first player wins if he has selected all edges of some complete subgraph on k vertices, see [2]. Let $N^*(k)$ be the least integer n so that the first player has a winning strategy, that is, the first player can always select all edges of some complete graph on k vertices. As proved by Erdős and Selfridge in [2] (the lower bound) and Beck in [1] (the upper bound), we have:

$$2^{\frac{k}{2}} < N^*(k) < (2+\epsilon)^k\,.$$

Generalizing the Ramsey game on pairs, Hahn and Širáň studied the following Ramsey-type game for graphs: Let G be a k-vertex graph, and let there be two players, the constructor and the destroyer. The players alternately pick previously unpicked edges of the complete graph on n vertices, and the constructor wins whenever he has selected all edges of some G, otherwise the destroyer is the winner, see [3].

Let G be a k-vertex star and let N^*_G be the least number of vertices on which the constructor has a winning strategy, that is, the constructor can always select all edges of some k-vertex star. In [3] it is proved:

$$1.2936k < N^*_G < 2k - \log_2 k\,.$$

In this paper we consider Ramsey-type games for spanning subgraphs of the complete graph on n vertices. We show that if $n \geq 5$ the constructor can always

Typeset by $\mathcal{AMS}$-TEX

construct a path on n vertices, and if $n \geq 15$ he can even construct a cycle on n vertices. (We suppose that the destroyer begins.) This can be interpreted as follows: If n satisfies the conditions mentioned above, the constructor can construct a Hamiltonian path (or a Hamiltonian cycle) in a complete graph on n vertices. Moreover, the constructor can construct a path or a cycle even if the destroyer has picked some, but at most $(n-5)/2$ or $(n-15)/2$ edges, respectively, before the game starts. Actually, our proofs will yield a certain class of trees on n vertices that can be constructed by the constructor.

2. Paths

In this section we consider a Ramsey-type game played on n vertices, where the constructor wins if and only if he has selected all edges of some path on n vertices. Let us denote this game by P_n^c (P_n^d) if the constructor (the destroyer) begins. We remark that the moves of the constructor will always be denoted by $c_1, c_2, \ldots$, while for the moves of the destroyer we use $d_1, d_2, \ldots$.

It is easy to see that the constructor wins the games P_2^c and P_3^c, while in P_4^c the destroyer is a winner (choosing d_1 nonadjacent to c_1, and d_2 nonadjacent to c_2). We prove here that in P_n^d, $n \geq 5$, the constructor is the winner, i.e., the destroyer loses even if he starts.

For the sake of convenience, if X and Y are two disjoint subsets of vertices, by $\langle X \rangle$ and $\langle Y \rangle$ we denote the set of edges having both endvertices in X and Y, respectively, and by XY we denote the set of edges having one endpoint in X and the other in Y. If no confusion is likely, an edge is often identified with the set of its endvertices.

Lemma 1. *The constructor wins both P_5^d and P_6^d.*

Proof. We utilize the fact that the constructor wins the games P_2^c and P_3^c.

Let c_1 be adjacent to d_1, and let d_2 be an arbitrary (previously unpicked) edge. It is easy to see that the vertex set can be partitioned into two sets, say X and Y, both of size at most 3, such that $d_1, d_2 \in XY$, and $c_1 \in \langle X \rangle$. Let us choose $c_2 \in \langle Y \rangle$ such that c_2 is adjacent to d_1.

For the moment consider the game P_5^d. We may assume that $|X| = 2$ and $|Y| = 3$. In what follows if $d_i \in \langle Y \rangle$, $i \geq 3$, then we choose $c_i \in \langle Y \rangle$, while if $d_i \in XY$ we pick $c_i \in XY$. Moreover, in the later case we choose c_i such that $c_i \cap Y \in \{\cup d_j : d_j \in XY, j \leq i\} \cap Y$ (observe that such a choice is always possible). As $d_1, d_2 \in XY$ and $c_1, c_2 \notin XY$, this choice requires that, when the game is finished, the constructor has joined all but one vertex from Y to X. Since he has paths on both X and Y, he has constructed a path on five vertices.

Now consider P_6^d. As $|X| = |Y| = 3$, we may assume that $d_3 \notin \langle Y \rangle$. Let us choose $c_3 \in \langle X \rangle$ such that (if possible) c_3 is adjacent to d_1 (if $d_3 \in XY$ then c_3 can surely be adjacent to d_1). Since the constructor has a path on X, in what follows only its endpoints are important. Let us denote the endpoints by X'. Now, in $\langle Y \rangle$ there is only one edge picked by the constructor, and in $X'Y$ there are at most two edges picked by the destroyer. (In the case $d_3 \in XY$ we have $d_1 \notin X'Y$ as all c_1, c_2 and c_3 are adjacent to d_1.) Hence, the constructor can proceed on X' and Y analogously as in the case of P_5^d.

When the game is finished the constructor has paths on both X and Y, and all but one vertex from Y he joined to X'. Hence, he constructed a path on six vertices. □

In the preceding proof, if the destroyer has picked d_i adjacent to the vertex from $X - X'$ (or made any useless move), then the constructor can make an arbitrary move. For this reason, in what follows we do not consider useless moves of the destroyer.

Theorem 2. *The constructor wins the game P_n^d if $n \geq 5$.*

Proof. By Lemma 1, we may assume $n > 6$.

Let us choose c_1 adjacent to d_1, and denote by X the vertices of c_1 and by Y the remaining $n-2$ vertices. By induction, the constructor has a winning strategy in P_{n-2}^d. Thus, if $d_i \in \langle Y \rangle$ then choose $c_i \in \langle Y \rangle$ according to this strategy, while if $d_i \in XY$ then pick $c_i \in XY$ such that d_i and c_i have a common vertex in Y whenever possible.

When the game is finished the constructor has a path on Y, and all but one vertex from Y he joined to X, i.e., he constructed a path on n vertices, as required. $\square$

One can see that the constructor's strategy is not as tight in the case $n > 6$ as in the case $5 \leq n \leq 6$. Namely, he can pause in the first occurrence of d_i in XY, $i \geq 2$. His first choice of $c_i \in XY$ is necessary when the destroyer has three edges in XY. Moreover, the constructor can avoid getting stuck at some disadvantageous vertices during the game analogously as in P_6^d.

Consider the following generalization of P_n^d: On an n-vertex set there is a subset B of k prescribed vertices, and the destroyer had picked l edges before the game started. In the game, the players alternately pick previously unpicked edges, the destroyer begins, and the constructor wins whenever he has selected all edges of some n-vertex path that does not have endpoints in B. Let us denote this game by $P_n^d(k, l)$.

Lemma 3. *If $n \geq 5 + 3k$ then the constructor wins the game $P_n^d(k, 0)$.*

Proof. If $k = 0$ then the constructor has a winning strategy in $P_n^d(0, 0)$ as $n \geq 5$, by Theorem 2. Suppose that $k \geq 1$ and let $b \in B$. We may assume that the destroyer had picked all edges from $\langle B \rangle$ before the game started.

Let us choose c_1 and c_2 both incident with b, and moreover, we choose c_1 adjacent to d_1 and, if d_2 is not adjacent to c_1, choose c_2 adjacent to d_2. (Observe that this is always possible.) Now let X be the set of endvertices of c_1 and c_2, and let Y be the set of remaining $n - 3$ vertices (i.e., Y contains $k - 1$ vertices from B). The constructor has a path on X, $d_1, d_2 \in XY$, and there are no picked edges in Y (except those in $\langle B - \{b\} \rangle$). Denote $X' = X - \{b\}$.

Clearly, $n - 3 \geq 5 + 3(k - 1)$. By induction, the constructor has a winning strategy in $P_{n-3}^d(k - 1, 0)$ on Y. Thus, if $d_i \in \langle Y \rangle$ then choose $c_i \in \langle Y \rangle$ according to this winning strategy, while if $d_i \in X'Y$ then choose $c_i \in X'Y$ such that $c_i \cap Y \in \{\cup d_j : d_j \in XY, j \leq i\} \cap Y$ whenever possible. The final condition requires that, when the game is finished, the constructor has constructed an n-vertex path that does not have endpoints in B. $\square$

Theorem 4. *If $n \geq 5 + 3k + 2l$ then the constructor wins the game $P_n^d(k, l)$.*

Proof. By Lemma 3, we may assume $l \geq 1$. We consider five cases **1. - 5.**, and in each of them we reduce the game $P_n^d(k, l)$ to $P_{n'}^d(k', l')$ such that $n' < n$ and $n' \leq 5 + 3k' + 2l'$. More precisely, after the first $j - 1$ moves of both players we split the n vertices into two sets X and Y, $|X| = j$ and $|Y| = n - j = n'$. The constructor will have a path on X (its endpoints we denote by X'), and the destroyer will have at most two edges in $X'Y$. In Y there will remain k' vertices from B and l' destroyer's edges, and the numbers n', k' and l' will satisfy the inequality mentioned above. By induction, the constructor has a winning strategy in $P^d_{n'}(k', l')$ on Y, and hence, next we pick $c_i \in \langle Y \rangle$ according to this strategy if $d_i \in \langle Y \rangle$, while if $d_i \in X'Y$ we pick $c_i \in X'Y$ such that $c_i \cap Y \in \{\cup d_j : d_j \in XY, j \leq i\} \cap Y$ whenever possible. This will result in the required n-vertex path.

Let D be the graph consisting of d_1 and the destroyer's l edges. Since $n \geq 5+3k+2l$, there is a set $F = \{f_1, f_2, \ldots\}$ of at least $3+2k$ vertices that are neither in B nor in D.

1. Suppose that there are two vertices of degree one in D, say u and v, such that uv is not in D.

Choose $c_1 = uv$. If $u, v \notin B$ then $X = X' = \{u, v\}$, $n' = n-2$, $k' = k$, $l' = l-1$, and $n-2 \geq 5+3k+2(l-1)$.

If $u, v \in B$ then choose $c_2 = f_1 u$, $c_3 = v f_2$. (It is not important if $d_2 = f_1 u$ as the set F is large enough, so that the constructor can choose another of its vertices. In what follows this fact will not be specifically mentioned.) Put $X = \{f_1, u, v, f_2\}$ and $X' = \{f_1, f_2\}$. Clearly, the destroyer has at most two edges in $X'Y$, $n' = n-4$, $k' = k-2$, $l' \leq l+1$, and $n-4 \geq 5+3(k-2)+2(l+1)$.

Finally, if $u \in B$ and $v \notin B$ choose $c_2 = f_1 u$, and put $X = \{f_1, u, v\}$, $X' = \{f_1, v\}$. (The case $u \notin B$ and $v \in B$ can be proved similarly.) We have $n' = n-3$, $k' = k-1$, $l' \leq l$, and $n-3 \geq 5+3(k-1)+2l$.

2. Suppose that there is a vertex, say u, of degree two in D.

Choose $c_1 = uf_1$. If $u \notin B$ then $X = \{u, f_1\}$ and $n-2 \geq 5+3k+2(l-1)$. If $u \in B$ choose $c_2 = f_2 u$, $X = \{f_2, u, f_1\}$, and $n-3 \geq 5+3(k-1)+2l$.

3. Suppose that there is a vertex, say u, of degree one in D. Since there are at least two edges in D, we may assume that there is a vertex, say v, of degree at least three in D such that uv is not in D, by **1.** and **2.**

Let $c_1 = uv$ and $c_2 = vf_1$. If $u \notin B$ we choose $X = \{u, v, f_1\}$ and $X' = \{uf_1\}$. The destroyer has at most two edges in $X'Y$, and $n-3 > 5+3k+2(l-2)$. If $u \in B$ choose $c_3 = f_2 u$, $X = \{f_2, u, v, f_1\}$, and $n-4 > 5+3(k-1)+2(l-1)$.

In the next cases we may assume that the degrees of the vertices in D are at least 3.

4. Suppose that u and v are vertices in D, each of degree at least three, and uv is not in D.

Choose $c_1 = uv$, $c_2 = f_1 u$, $c_3 = vf_2$, and put $X = \{f_1, u, v, f_2\}$ and $X' = \{f_2, f_1\}$. The destroyer has at most two edges in $X'Y$ and $n-4 > 5+3k+2(l-3)$.

5. Suppose that D is a complete graph on at least four vertices.

Let u be a vertex of degree at least three in D. Choose $c_1 = uf_1$, and $c_2 = vu$ such that v is not in D and d_2 is adjacent to c_1 or c_2. (This is possible as D is a complete graph.) If $v \notin B$ then $X = \{v, u, f_1\}$, $X' = \{v, f_1\}$, the destroyer has at most one edge in $X'Y$, and $n-3 > 5+3k+2(l-2)$. If $v \in B$ choose $c_3 = f_2 v$, $X = \{f_2, v, u, f_1\}$, and $n-4 > 5+3(k-1)+2(l-1)$. $\square$

3. Cycles

In this section we consider a Ramsey-type game played on n vertices, where the constructor wins if and only if he has selected all edges of some cycle on n vertices. We denote this game by R^c_n (R^d_n) if the constructor (the destroyer) begins.

The constructor loses in R^c_n if $n \leq 4$, since the cycle has too many edges. Moreover, he loses in R^c_5 (choose d_1 nonadjacent to c_1, d_2 adjacent to d_1, and d_3 such that $\{d_1, d_2, d_3\}$ is either a 3-cycle or contains a vertex of degree three), and in R^c_6 (choose d_1 nonadjacent to c_1; the constructor likes to pick at least two edges incident with each vertex, and utilizing this fact in the first five moves the destroyer can pick $K_4 - e$, i.e., a complete graph on four vertices without one edge). However, for $n \geq 15$ we have:

Theorem 5. *The constructor wins the game R^d_n if $n \geq 15$.*

Proof. In the first five moves the constructor picks a 4-cycle, and then a path on remaining $n - 4$ vertices. Since the endvertices of the path will be joined to the 4-cycle in a good way, this will result to a cycle on n vertices.

Let us choose $c_1 = yx$ adjacent to d_1 (assume that d_1 is incident with y). Moreover, choose $c_2 = yz$ adjacent to d_2. (If d_1, d_2 and c_1 form a triangle, choose any $c_2 = yz$.) Then c_1 and c_2 form a path on three vertices. As $n > 7 + 2$ we may choose c_3 nonadjacent to any of the previously picked edges. Let $c_3 = uv$. It is easy to see that no matter how the destroyer moves, we may choose $c_4 \in \{y\}\{u, v\}$, say $c_4 = yu$, and then $c_5 \in \{v\}\{x, z\}$, say $c_5 = vx$, to obtain a 4-cycle (in this case $(xyuv)$). We remark that if $d_5 \notin \{v\}\{x, z\}$, then there are two possibilities for c_5, namely vx and vz, and we prefer that one for which d_2 and c_5 are adjacent.

Let $X_1 = \{x, u\}$, $X_2 = \{y, v\}$, $X = X_1 \cup X_2$, and let Y be the set of the remaining $n - 4$ vertices. In XY there are at least two destroyer's edges (either d_1 and d_2 or, if $d_5 \in \{v\}\{x, z\}$, d_1 and d_5). Split XY into pairs of edges $\{X_1\{a\}, X_2\{a\} : a \in Y\}$. Denote by A' those pairs in which the destroyer has picked an edge in the first five moves. In what follows we define a set $A = \{X_{i_1}\{a_1\}, X_{i_2}\{a_2\}, \ldots, X_{i_m}\{a_m\}\}$, $i_1, \ldots, i_m \in \{1, 2\}$. If there is $X_j\{a\} \in A'$, $1 \leq j \leq 2$, with both edges picked by the destroyer, then let both edges $X_{i_1}\{a_1\}$ and no edge of $X_{i_2}\{a_2\}$ are picked by the destroyer and in this case we set $A = A' \cup \{X_{i_2}\{a_2\}\}$. Otherwise $A = A'$. Note that in either case there are exactly two destroyer's edges in $X_{i_1}\{a_1\}$ and $X_{i_2}\{a_2\}$, $2 \leq m \leq 5$, and there are at most five destroyer's edges in $X_{i_j}\{a_j\}$, $1 \leq j \leq m$.

From the sixth move on we will use the following strategy:

1. If $d_i \in \langle Y \rangle$, choose $c_i \in X_{i_j}\{a_j\}$, $3 \leq j \leq m$.

2. If $d_i \in X_j\{a\}$, $1 \leq j \leq 2$, such that $X_j\{a\} \notin A$, then choose $c_i \in X_j\{a\}$.

3. If $d_i \in X_{i_j}\{a_j\}$, $1 \leq j \leq 2$, then choose $c_i \in X_{i_{j'}}\{a_{j'}\}$, $1 \leq j' \leq 2$.

4. If $d_i \in X_{i_j}\{a_j\}$, $3 \leq j \leq m$, then choose $c_i \in X_{i_{j'}}\{a_{j'}\}$ such that $3 \leq j' \leq m$ whenever possible.

We will proceed using this strategy until both edges are picked (by any of the players) in all $X_{i_j}\{a_j\}$, $3 \leq j \leq m$. (This will happen as the game is finite.)

Thus, we may assume that there are no unpicked edges in $X_{i_j}\{a_j\}$, $3 \leq j \leq m$. Let B consist of those a_j, $3 \leq j \leq m$, for which the destroyer has picked both edges of $X_{i_j}\{a_j\}$, $|B| = k$, and let l be the number of the destroyer's edges in $\langle Y \rangle$. In what follows, the constructor will play $P^d_{n-4}(k, l)$ on Y. There are three cases possible:

1. B is empty. In this case $l \leq 3$ (as two from the destroyer's first five edges are in $X_{i_1}\{a_1\}$ and $X_{i_2}\{a_2\}$). By Theorem 4 if $n - 4 \geq 5 + 3 \cdot 2$ the constructor wins the game $P^d_{n-4}(0, l)$.

2. $|B| = 1$. Then $l \leq 1$ and the constructor wins $P^d_{n-4}(1, l)$ if $n - 4 \geq 5 + 1 \cdot 3 + 1 \cdot 2$, by Theorem 4. (If $d_i \in X_{i_j}\{a_j\}$, $3 \leq j \leq m$, was the final edge chosen by the destroyer and it was not possible to choose $c_i \in X_{i_{j'}}\{a_{j'}\}$, $3 \leq j' \leq m$, then we can choose $c_i \in \langle Y\rangle$ according to the winning strategy for $P^d_{n-4}(1,l)$, $l \le 1$, where the destroyer has already picked its first edge.)

3. $|B| = 2$. In this case $l = 0$, and the constructor wins $P^d_{n-4}(2,0)$ if $n-4 \ge 5+2\cdot 3$, by Theorem 4.

Since $n \ge 15$, in all three cases the constructor wins $P^d_{n-4}(k,l)$, i.e., he can construct an $(n-4)$-vertex path on Y whose endvertices are not in B.

Now proceed in our game: If $d_i \in \langle Y\rangle$ choose $c_i \in \langle Y\rangle$ according to the winning strategy for $P^d_{n-4}(k,l)$, while if $d_i \in X_j\{a\}$, $1 \le j \le 2$, choose $c_i \in X_j\{a\}$. (In the case $d_i \in X_{i_j}\{a_j\}$, $1 \le j \le 2$, choose $c_i \in X_{i_{j'}}\{a_{j'}\}$, $1 \le j' \le 2$.)

When the game is finished, there is a 4-cycle on X and an $(n-4)$-vertex path P on Y that does not have endvertices in B. Let e_1 and e_2 be the endvertices of P. Our strategy requires that at most one from $X_1\{e_1\}$, $X_2\{e_1\}$, $X_1\{e_2\}$, $X_2\{e_2\}$ has both edges picked by the destroyer, say $X_1\{e_1\}$ (in this case $e_1 = a_1$ or $e_1 = a_2$). Thus, there are constructor's edges in both $X_2\{e_1\}$ and $X_1\{e_2\}$, and these edges together with three edges of the 4-cycle $(xyuv)$ and the edges of P form an n-vertex cycle, i.e., the constructor has won. $\square$

We remark that $n \ge 15$ is our best estimate even for R^c_n, since the destroyer can choose $d_1 = wz$ and $d_2 = wx$ in the preceding proof and three edges from $d_1, \dots, d_4$ will be in $\langle Y\rangle$.

Let $R^d_n(l)$ be a Ramsey-type game where the constructor wins if and only if he has selected all edges of some n-vertex cycle, however, the destroyer (who begins) had picked l edges before the game started. We have:

Theorem 6. *If $n \ge 15 + 2l$ then the constructor wins the game $R^d_n(l)$.*

The proof is similar to that of Theorem 5. The only difference is that there will be l more edges in $\langle Y\rangle$ and applying Theorem 4 we obtain the result.

4. Trees

Let T be a prescribed n-vertex tree. By T^d_n we denote a Ramsey-type game played on n vertices, where the destroyer begins and the constructor wins if and only if he has selected all edges of some T.

Let T be a tree. Suppose that the edge set of T can be decomposed into a subtree T_0 (having l edges) and a nonempty collection of paths, say $P_1, \dots, P_k$, that may pairwise intersect only in the vertices of T_0. If each of the paths contains at least $15 + 2\lceil \frac{l}{k} \rceil$ vertices, we write $T \in \mathcal{T}$.

In this section we show that if $T \in \mathcal{T}$ then the constructor is a winner in T^d_n.

Lemma 7. *Let G be a graph on $k(m-1) + k'$ vertices with l edges, $1 \le k' \le k$. Let $X = \{x_1, \dots, x_{k'}\}$ be some vertices of G, and let Y be the set of the remaining $k(m-1)$ vertices. Moreover, let $k_1 + \dots + k_{k'} = k$ and $k_i \ge 1$, $1 \le i \le k'$. Then there are vertex sets $X_1, \dots, X_k$ each of size m, such that $|X_j \cap X| = 1$, $X_1 - X$, $X_2 - X$, $\dots$, $X_k - X$ is a partition of Y, x_i is in k_i of the X_j's, and each $\langle X_j\rangle$ contains at most $\lceil \frac{l}{k} \rceil$ edges, $1 \le j \le k$ and $1 \le i \le k'$.*

Proof. Let $X^2, X^3, \dots, X^m$ be a partition of Y, $|X^2| = \dots = |X^m| = k$, such that there is m^0, $1 \le m^0 \le m$, for which if $x' \in X^j$, $2 \le j < m^0$, then $X\{x'\}$ is nonempty, while if $x' \in X^j$, $m^0 < j \le m$, then $X\{x'\}$ is empty (moreover, if $m^0 > 1$, we may assume that XX^{m^0} is not empty). Let $X^1_1 = X^1_2 = \dots = X^1_{k_1} = \{x_1\}$,

$\ldots, X^1_{k_{k'-1}+1} = \cdots = X^1_{k_{k'}} = \{x_{k'}\}$. We construct $X_i^2, \ldots X_i^m$ such that $|X_i^j| = j$, $\cup_{i=1}^k (X_i^j - X_i^{j-1}) = X^j$, and if $\langle \cup_{i=1}^k (X_i^j - X)\rangle$ contains l_j edges, then $\cup_{i=1}^k \langle X_i^j - X\rangle$ will have at most $\frac{l_j}{k}$ edges, $1 \le j \le m$.

By induction, suppose that this is true for all j', $2 \le j' \le j$. Let l' be the number of edges in $(\cup_{i=1}^k (X_i^j - X))X^{j+1}$. We construct X_i^{j+1} from X_i^j such that $\cup_{i=1}^k (X_i^{j+1} - X_i^j) = X^{j+1}$. Clearly, there are $k!$ possibilities for constructing X_i^{j+1}'s in this way. Let e be an edge in $(X_i^j - X)X^{j+1}$. In $(k-1)!$ cases we have $e \in \langle X_i^{j+1}\rangle$. Thus, the average number of new edges in $\cup_{i=1}^k \langle X^{j+1} - X\rangle$ is $\frac{l'(k-1)!}{k!} = \frac{l'}{k}$. Hence, the required sets X_i^{j+1}, $1 \le i \le k$, exist such that $\cup_{i=1}^k \langle X^{j+1} - X\rangle$ contains at most $\frac{l_j}{k}$ edges. Thus, there are at most $\frac{l_m}{k}$ edges in each $\langle X_i^m - X\rangle$.

However, there are still l'' edges in XY, $l'' > k(m^0 - 2)$, and in each $\langle X_i^m\rangle$ we have at most $m^0 - 1$ from these l'' edges. As $m^0 - 1 < \frac{l''}{k} + 1$, there are at most $\lceil \frac{l}{k} \rceil$ edges in each $\langle X_i^m\rangle$, $1 \le i \le k$. $\square$

Theorem 8. *The constructor wins the game T_n^d if $T \in \mathcal{T}$.*

Proof. As $T \in \mathcal{T}$, it consists of a subtree T_0 and k paths $P_1, \ldots, P_k$. In the first l moves the constructor will construct T_0. (Recall that l is the number of edges of T_0.) This can be done step by step by joining a new vertex (that is not incident with the destroyer's edges) to the subtree of T_0 just constructed. When T_0 is constructed, there are l edges picked by the destroyer.

Let $X = \{x_1, \ldots, x_{k'}\}$ be the vertices of both T_0 and $\cup_{i=1}^k P_i$, each x_i lying on k_i paths from $P_1, \ldots, P_k$, and let Y' be the set of vertices that are not in T_0 ($|Y'| = n - l - 1$). Moreover, let Y be a subset of Y', $|Y| = 2k\lceil \frac{l}{k} \rceil$, such that there are no picked edges incident with vertices in $Y' - Y$. (Observe that $2l \le 2k\lceil \frac{l}{k} \rceil < n - l - 1$.) Let $X_1, \ldots, X_k$ be the sets whose existence is guaranteed by Lemma 7 ($\cup_{i=1}^k X_i = X \cup Y$). Then there are at most $\lceil \frac{l}{k} \rceil$ edges in each X_i, $1 \le i \le k$.

Now extend every X_i to X_i^* by adding some of those vertices from $Y' - Y$ that are not in X_j^*, $j < i$, and do this so that X_i^* will have as many vertices as P_i. Since $|X_i^*| \ge 15 + 2\lceil \frac{l}{k} \rceil$ and there are at most $\lceil \frac{l}{k} \rceil$ destroyer's edges in $\langle X_i^*\rangle$, $1 \le i \le k$, by Theorem 6 the constructor has a winning strategy in $R^d_{|X_i^*|}(\lceil \frac{l}{k} \rceil)$ on X_i^*. Thus, if $j > l$ and $d_j \in \langle X_i^*\rangle$, $1 \le i \le k$, choose $c_j \in \langle X_i^*\rangle$ according to this winning strategy to obtain T. $\square$

Let T consist of a star T_0 and a path P_1 such that T_0 has $l = \lfloor \frac{n-15}{3} \rfloor$ edges, P_1 has $n - l$ vertices ($n - l \ge 15 + 2l$), and P_1 crosses T_0 in the central vertex. Then $T \in \mathcal{T}$ and the maximum degree in T equals $l + 2 = \lfloor \frac{n-9}{3} \rfloor$. Thus, the constructor can win in T_n^d even if T contains a vertex of degree $\lfloor \frac{n-9}{3} \rfloor$.

Acknowledgement

I would thank to J. Bukor and J. Širáň for their interest in this work and helpful comments.

References

[1] Beck J., *Van der Warden and Ramsey type games*, Combinatorica **1** (1981), 103-116.

[2] Erdős P., Selfridge J. L., *On a combinatorial game*, J. Comb. Theory (A) **14** (1973), 298-301.

[3] Hahn G., Širáň J., *Star wars*, preprint (1995).