CHARACTERIZATION OF MINOR-CLOSED PSEUDOSURFACES

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Abstract. A pseudosurface is obtained from a collection of closed surfaces by identifying some points. It is shown that a pseudosurface S is minor-closed if and only if S consists of a pseudosurface S°, having at most one singular point, and some spheres glued to S° in a tree structure.

1. Introduction

By a *pseudosurface* we understand a connected topological space resulting when finitely many identifications, of finitely many points each, are made on a finite collection of closed surfaces (=compact 2-manifolds). Any point obtained by such an identification of at least two distinct points is called a *singular point*. Let G be a graph and S a surface or a pseudosurface. We say that G is *embeddable in* S if there is a continuous mapping $\varphi : G \to S$ which maps G homeomorphically onto its image $\varphi(G)$. An embedding $\varphi : G \to S$ is called a *2-cell (or cellular) embedding* if each component of $S - \varphi(G)$, called a *face*, is homeomorphic to an open 2-cell.

Embeddability in a closed surface can be characterized by a finite set of *forbidden subgraphs*; constructive proofs of this theorem were given by Kuratowski for the sphere [4], Bodendiek and Wagner for orientable surfaces [3], and by Archdeacon and Huneke for nonorientable surfaces [1]. It is natural to ask whether the same is true for pseudosurfaces. The answer is negative in general, as shown by Širáň and Gvozdjak in [7] for 2-banana surface, i.e. the 2-amalgamation of two spheres. However, the 2-banana surface is not minor-closed, see [2]. We remark that a surface S is minor-closed if and only if the set of graphs embeddable in S is minor closed (i.e. closed under a deletion of an edge or a vertex, and under contraction of an edge).

As shown by Robertson and Seymour in [6], any minor-closed class of graphs can be characterized by a finite set of forbidden subgraphs. Thus, it seems to be reasonable to characterize minor-closed pseudosurfaces; by [6] the embeddability in such pseudosurfaces can be characterized by a finite set of forbidden subgraphs.

Let S be a pseudosurface. If S contains as a topological subspace a sphere S_1 having exactly one singular point, then S is called *spherically-reducible*. Otherwise, S is called *spherically-irreducible*. Clearly, from each pseudosurface S we obtain a spherically-irreducible pseudosurface S° by successively deleting the spheres that are "glued" to the rest of the pseudosurface in exactly one singular point. Moreover, S° is determined uniquely by S. The main result of this paper is the following theorem:

THEOREM 1. *Let S be a pseudosurface. Let S° be the spherically-irreducible pseudosurface that arises from S by successively deleting the spheres containing exactly one singular point. Then S is minor-closed if and only if S° contains at most one singular point.*

2. Preliminaries

Let G be a graph. As usual, $V(G)$ denotes the vertex set of G and $E(G)$ the edge set of G. The degree of a vertex u in G is denoted by $deg_G(u)$. By G/uv we denote a graph that arises from G by contracting the edge $uv \in E(G)$. A cycle on n vertices is denoted by C_n and a path on n vertices is denoted by P_n.

Let S be a closed surface and let G be a graph cellularly embedded in S with F faces. Then the number

$$\chi(S) = |V(G)| - |E(G)| + F$$

depends only on S (and not on G) and is known as the *Euler characteristic* of S. The non-negative quantity $\epsilon(S) = 2 - \chi(S)$ is called the *Euler genus* of S. If S is orientable, then S has a *positive orientability characteristic*. Otherwise, S has a *negative orientability characteristic*. We remark that S is determined uniquely by $\epsilon(S)$ and the orientability characteristic.

Definitions and notations not included here can be found in White [8].

In what follows we introduce concepts of uniqueness and faithfulness due to [5].

Two embeddings $\varphi_1, \varphi_2 : G \to S$ are said to be *equivalent* if there is an automorphism $\sigma : G \to G$ and a self-homeomorphism $h : S \to S$ with $h \circ \varphi_1 = \varphi_2 \circ \sigma$. When there is just one equivalence class of embeddings of G in S, G is said to be *uniquely embeddable* in S.

Faithfulness is defined as follows. Let $\varphi : G \to S$ be an embedding of G in S. Then φ is said to be *faithful* if for any automorphism $\sigma : G \to G$, there is a self-homeomorphism $h : S \to S$ such that $h \circ \varphi = \varphi \circ \sigma$. In other words, φ is faithful when all automorphisms of $\varphi(G)$ extend to self-homeomorphisms of S. A graph G is said to be *faithfully embeddable* in S if G has a faithful embedding in S.

Thus, G is uniquely and faithfully embeddable in S if G has a unique embedding in S and this embedding is faithful.

For an arbitrary closed surface S there exists a graph uniquely and faithfully embeddable in S by the following lemma [5, Proposition 1.4.7]:

LEMMA 1. *Every closed surface admits an infinite number of triangulations that are uniquely and faithfully embeddable in it.*

Let G be uniquely and faithfully embeddable in S. Then G is uniquely embeddable in S as a labeled graph. Consider a faithful embedding of G in S. Then no automorphism of G can map a vertex u of G again to u and rearrange the neighbors of u. This local property of faithful embedding will often be tacitly used.

Let G be uniquely and faithfully embeddable in S, and let H be a subdivision of G. Then obviously, H is again uniquely and faithfully embeddable in S. Moreover, we have the following lemma [5, Corollary 1.5.7]:

LEMMA 2. *Let G have a unique and faithful triangular embedding in a closed surface S. If a 3-connected graph H is embeddable in S and contains a subgraph contractible to G, then H is uniquely and faithfully embeddable in S.*

In the proof of Theorem 1 we use the following lemma:

LEMMA 3. *Every closed surface S admits infinitely many triangulations that are uniquely and faithfully embeddable in S, and that cannot be embedded in S' with $\epsilon(S') = \epsilon(S)$ and the opposite orientability characteristic.*

Proof. Let G be uniquely and faithfully triangularly embeddable in S. In what follows we construct the barycentric subdivision G_2 of G and show that G_2 satisfies the conditions in Lemma 3. First subdivide all edges of G by one vertex and denote the resulting graph by G_1. Clearly, G_1 has a 2-cell embedding, say φ_1, in S. Now insert one new vertex into each face f of φ_1, join it to all vertices lying on the boundary of f, and denote the resulting graph by G_2.

Since G triangulates S and contains no loops, there are no multiple edges in G_2. Since G_2 is 3-connected and contains a subgraph contractible to G, G_2 is uniquely and faithfully embeddable in S, by Lemma 2. Moreover, the unique embedding of G_2 in S is a triangulation of S.

Now assume that G_2 is embedded in S' with $\epsilon(S') = \epsilon(S)$. Then G_2 necessarily triangulates S'. Clearly, each 3-cycle in G_2 contains exactly one vertex from $V(G)$, one vertex from $V(G_1)-V(G)$, and one vertex from $V(G_2)-V(G_1)$. Moreover, each edge of G_2 lies in exactly two 3-cycles. Thus, the surface admitting a triangular embedding of G_2 is determined uniquely, and hence $S \cong S'$.

By Lemma 1 there are infinitely many triangulations of S satisfying Lemma 3. $\square$

As a matter of fact, the graphs satisfying Lemma 1 were constructed from triangulations by means of barycentric subdivision, see [5]. Hence, they also satisfy Lemma 3.

3. Proof of the main result

This section is completely devoted to the proof of Theorem 1.

Proof. Let S be a pseudosurface, and let S° be the spherically-irreducible pseudosurface that arises from S by successively deleting the spheres containing exactly one singular point. Suppose that S° contains at most one singular point.

Clearly, each pseudosurface is closed under deletion of an edge or a vertex. Thus, it is sufficient to prove that S is closed under edge contraction.

Let G be a graph embeddable in S, and let φ be an embedding of G in S. Then the subgraph of G embedded in $S - S^\circ$ in φ is planar. Thus, G is embeddable in S°. Clearly, S° is closed under edge contraction, and hence, S is minor-closed.

We now turn to the more difficult part of Theorem 1. The outline of the proof is as follows. Suppose that S is a pseudosurface closed under edge contraction. We construct a graph G embeddable in S with two specified vertices z_1 and z_2 that are joined by an edge. Then we derive properties (*i*) - (*iv*) of any embedding φ^c of G/z_1z_2 in S. Finally, considering various positions of z_1 and z_2 in G on S, and using (*i*) - (*iv*) we obtain assertions (1) - (4) that complete the proof.

Let S be a pseudosurface resulting when identifications are made on a collection $S_1, S_2, \ldots, S_l$ of closed surfaces. For the sake of convenience, with S we associate a bipartite multigraph B_S. The vertex set of B_S consists of S_i, $1 \le i \le l$, and the set P of singular points of S, and S_i is joined to $p \in P$ by t edges if and only if t points of S_i have been identified to p. Denote $n = |P|$, and $n_i = deg_{B_S}(S_i)$, $1 \leq i \leq l$.

Let p be a singular point of S. If there is S_i, $1 \leq i \leq l$, that is joined to p by at least two edges in B_S, then p is called a *self-singular* point. By S_i^* we denote the topological subspace of S, which had been obtained from S_i, $1 \leq i \leq l$. More precisely, $B_{S_i^*}$ is a subgraph of B_S induced by multiple edges incident with S_i.

The construction of G

By Lemma 3 there are graphs H_i uniquely and faithfully triangularly embeddable in S_i, $1 \leq i \leq l$, which cannot be embedded in S_i' with $\epsilon(S_i') = \epsilon(S_i)$ and the opposite orientability characteristic. We can assume that each H_i has at least n_i vertices.

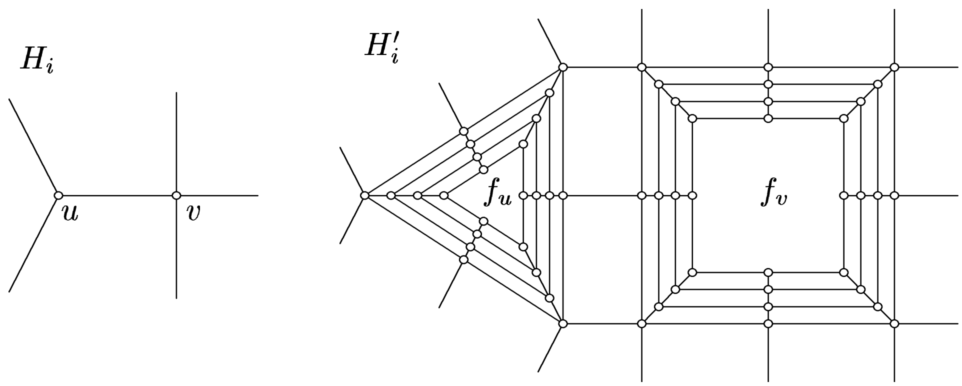

Figure 1

Now we locally describe a construction of a graph H_i' from H_i, $1 \leq i \leq l$. We replace each vertex u of H_i by the Cartesian product $C_{n \cdot deg_{H_i}(u)} \times P_{n+2}$ and each edge by $n+1$ independent edges as shown in Fig. 1 for $n = 2$. Clearly, H_i' is 3-connected, embeddable in S_i, and contains a subgraph contractible to H_i. Thus, by Lemma 2 H_i' is uniquely and faithfully embeddable in S_i, and the only embedding φ_i' of H_i' in S_i is just the one locally described above.

For every $u \in V(H_i)$ denote by f_u the face of the embedding φ_i' that appears in the position of u in S_i, $1 \leq i \leq l$ (see Fig. 1). For a moment we concentrate on H_1. Put one new vertex u' into each face f_u of φ_1', and join u' to all vertices incident with f_u. There are at least n_1 such added vertices u'; out of them we need to distinguish n_1-1 vertices, say $v_2^1, \ldots, v_{n_1}^1$. Moreover, put one new vertex v_1^1 into the face where v_2^1 has been placed, join v_1^1 to v_2^1, and denote the resulting graph by G_1, see Fig. 2. Similarly, for each i, $2 \leq i \leq l$, put one new vertex u' into each face f_u of

φ_i', join u' to all vertices incident with f_u, and denote the resulting graph by G_i. Denote by $v_1^i, \ldots, v_{n_i}^i$ the n_i vertices of $V(G_i) - V(H_i')$, $2 \le i \le l$.

Finally, identify $v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^l, \ldots, v_{n_l}^l$ into n vertices $z_1, \ldots, z_n$ in the same way as the corresponding points of $S_1, \ldots, S_l$ have been identified when constructing the pseudosurface S, and denote the resulting graph by G. More precisely, there is a one-to-one correspondence between the vertices v_j^i, $1 \le i \le l$ and $1 \le j \le n_i$, and the edges of B_S incident to S_i. Identify v_j^i with $v_{j'}^{i'}$ whenever the corresponding edges of B_S are incident to the same singular point. Note that the structure of G depends on the ordering of the surfaces $S_1, \ldots, S_l$ and the singular points of S. However, the assertions (i) - (iv) we are going to prove below do not depend on this ordering.

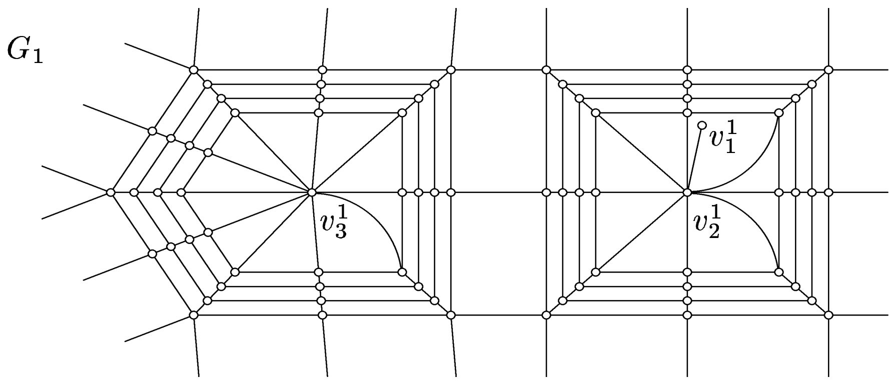

Figure 2

Clearly, G is embeddable in S. Denote by φ the embedding of G in S which is determined by the embeddings φ_i', $1 \le i \le l$, as described above.

Denote by z_1 and z_2 the vertices of G obtained from v_1^1 and v_2^1, respectively. Assume that $z_1 \ne z_2$. Suppose that G/z_1z_2 is embeddable in S and denote by φ^c an embedding of G/z_1z_2 in S. In what follows we derive some properties of φ^c. (We remark that so far we have not had any reason to expect that the supposed embedding φ^c of G/z_1z_2 in S has anything in common with the original embedding φ of G in the same S.)

Basic properties of φ^c

There are $m \le n$ vertices, say $x_1, x_2, \ldots, x_m$, of G/z_1z_2 embedded in the singular points of S in φ^c. Let H' be a subgraph of G/z_1z_2. If H' contains no vertex from $\{x_1, \ldots, x_m\}$, then H' is called an *unbroken* subgraph of G/z_1z_2.

For each i, $1 \le i \le l$, let us do the following. Find a connected subgraph H_i'' of $H_i' - \{x_1, \ldots, x_m\}$ that is uniquely and faithfully embeddable in S_i.

For each $u \in V(H_i)$, include to H_i'' all unbroken copies of $C_{n \cdot deg_{H_i}(u)}$ at u. (Since $m \leq n$, for each $u \in V(H_i)$ there are at least two copies of $C_{n \cdot deg_{H_i}(u)}$ in H_i''.) Moreover, for each $uv \in E(H_i)$, include to H_i'' all those unbroken copies of P_{n+2} at u that correspond to the unbroken copies of P_{n+2} at v, together with the edges joining them. (Since $m \leq n$, for each edge $uv \in E(H_i)$ there is a pair of corresponding copies of P_{n+2} in H_i''.) Finally, throw away the endvertices of H_i'', see Fig. 3.

Clearly, H_i'' contains a subgraph contractible to H_i. Since H_i'' is a subgraph of H_i', H_i'' is embeddable in S_i. Moreover, from H_i'' we obtain a 3-connected graph by a successive contraction of edges incident with vertices of degree two. Thus, H_i'' is uniquely and faithfully embeddable in S_i, by Lemma 2 and the note before Lemma 2.

Note that each H_i'' is embedded in one closed surface, say S_{i^c}, in φ^c, since the connected graph H_i'' contains no vertices placed in singular points. Clearly, $\epsilon(S_i) \leq \epsilon(S_{i^c})$, since H_i triangulates S_i and H_i'' contains a subgraph contractible to H_i. Let $J_t = \{j : \epsilon(S_j) > t\}$, $t \geq 0$. Assume that there is t such that $S_{j^c} \cong S_j$ for each $j \in J_t$ (this is certainly true for t large enough). Let $j \in J_t$. Then φ^c induces a cellular embedding of H_j'' in S_{j^c}. Thus, only planar graphs can be embedded in S_{j^c} together with H_j''. By the finiteness of J_t, for each $j \in J_t$ there is $k \in J_t$ such that $S_{k^c} = S_j$ (since $t \geq 0$, H_k'' is not a planar graph).

Suppose that $\epsilon(S_i) = t$. If S_i is not a sphere, then H_i'' is not a planar graph, and hence $i^c \notin J_t$. Thus, $\epsilon(S_{i^c}) \leq \epsilon(S_i)$ and hence $\epsilon(S_{i^c}) = \epsilon(S_i)$. Moreover, since H_i is not embeddable in S' with $\epsilon(S') = \epsilon(S_i)$ and the opposite orientability characteristic, we have $S_{i^c} \cong S_i$. Hence, $S_{j^c} \cong S_j$ for each $j \in J_{t-1}$. Thus:

(i) *For each i, $1 \leq i \leq l$, φ^c induces an embedding of H_i'' in S_{i^c}, $1 \leq i^c \leq l$. If S_i is not a sphere, we have $S_{i^c} \cong S_i$. Moreover, if $i_1 \neq i_2$, and S_{i_1} and S_{i_2} are not spheres, then $S_{i_1^c} \neq S_{i_2^c}$.*

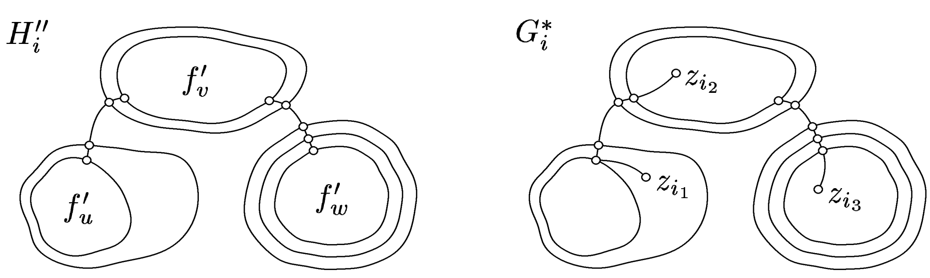

Figure 3

Clearly, each vertex from $V(G_i) - V(H_i')$ (except v_1^1) is joined to H_i'' by at least $n+1$ vertex-disjoint paths in G_i. Since v_1^1 and v_2^1 are identified into a single vertex in G/z_1z_2, we have:

(*ii*) *All vertices of G/z_1z_2 that have been obtained by the identification of some vertices from $V(G_i)-V(H_i')$, and possibly some other vertices, lie in $S_{i^c}^*$ in φ^c, $1 \leq i \leq l$.*

Suppose that S_i is a sphere, but S_{i^c} is not. Then there is a j such that S_j is not a sphere and $S_{j^c} = S_{i^c}$, by (*i*) (the second part). Moreover, φ^c induces a cellular embedding of H_j'' in S_{i^c}, and hence, H_i'' is embedded in one cell of the embedding of H_j'' in S_{i^c}. Thus, φ^c induces an embedding of H_i'' in S_{i^c} that arises from the embedding of H_i in S_i, since H_i'' is uniquely and faithfully embeddable in the sphere (we do not distinguish the exterior face of the embedding of H_i'' in the cell). Analogously, if S_i is not a sphere, or if S_{i^c} is a sphere, then φ^c induces an embedding of H_i'' in S_{i^c} that arises from the embedding of H_i in S_i, by (*i*).

Let $V_i = \{v_1^i, \ldots, v_{n_i}^i\} - \{v_1^1\}$. Denote by f_u' the face of the embedding of H_i'' in S_{i^c} that corresponds to the face f_u in φ_i', see Fig. 3. Since $m \leq n$, for all pairs $u, v \in V(H_i)$ there are at least four vertex-disjoint cycles in the embedding of H_i'' in S_{i^c} that separate f_u' from f_v', namely the copies of $C_{n \cdot deg_{H_i}(u)}$ and $C_{n \cdot deg_{H_i}(v)}$. Suppose that $u, v \in V_i$ have been identified to z in G/z_1z_2. Since there are at least two vertex-disjoint cycles separating u from v in S_{i^c} in φ^c (the exterior ones, see Fig. 3), z is placed in a self-singular point of $S_{i^c}^*$ in φ^c. Analogously, we have:

(*iii*) *Let $v_1, \ldots, v_a \in V_i$ be identified to a vertex z in G/z_1z_2, $a \geq 2$. Then z is placed in φ^c in a self-singular point p of $S_{i^c}^*$ that is joined to S_{i^c} by at least a edges in B_S.*

Now we introduce a lexicographical ordering of pseudosurfaces $S_{i'}^*$ for which $S_{i'} \cong S_i$, according the multiplicities of edges in $B_{S_{i'}^*}$. Let $S_{i_1} \cong S_{i_2}$. Let $S_{i_k}^*$ contain b_k self-singular points with multiplicities (i.e. the multiplicities of edges in $B_{S_{i_k}^*}$) $a_1^k \geq \cdots \geq a_{b_k}^k$, $1 \leq k \leq 2$. We write $S_{i_1}^* \preceq S_{i_2}^*$ if and only if from $a_j^1 > a_j^2$, $1 \leq j \leq b_1$ it follows that there is j', $1 \leq j' < j$, with $a_{j'}^1 < a_{j'}^2$. If $S_{i_1}^* \preceq S_{i_2}^*$ and $S_{i_1}^* \ncong S_{i_2}^*$, we write $S_{i_1}^* \prec S_{i_2}^*$.

Let z be a vertex of G/z_1z_2 that has been obtained by the identification of a vertex from V_i, and possibly some other vertices. Denote by P_z the collection of the paths joining z to H_i'' that contain no vertex from $\{x_1, \ldots, x_m\}$ (except possibly z). Clearly, for each such z there is at least one path in P_z with this property. Denote by G_i^* the subgraph of G/z_1z_2 induced by H_i'' and the paths P_z, where z is obtained by the identification of a vertex from V_i, see Fig. 3 (the vertices z_{i_j} in Fig. 3, $1 \leq j \leq 3$, need not necessarily be distinct). Since H_i'' is embedded in S_{i^c} in φ^c, the graph G_i^* is embedded in $S_{i^c}^*$, by (*ii*).

Suppose that S_i is not a sphere. Then $S_{i^c} \cong S_i$, by (*i*). Moreover, we have $S^*_{i^c} \succeq S^*_i$, by (*ii*) and (*iii*). Note that $S^*_{1^c} \succeq S^*_1$ also if z_1 and z_2 are self-singular points of S^*_1. (We remark that $S^*_{i^c} \succeq S^*_i$ is only a necessary but not a sufficient condition for embeddability of G^*_i in $S^*_{i^c}$). Let $J = \{j : S^*_j \succ S^*_i\}$. By (*i*) (the second part), if $k_1 \neq k_2$, and $k_1, k_2 \in J$, then $S_{k^c_1} \neq S_{k^c_2}$. Since J is a finite set, $S^*_{i^c} \succ S^*_i$ contradicts $S^*_{j^c} \succeq S^*_j$, $j \in J$. Hence, $S^*_{i^c} \cong S^*_i$.

Now suppose that S_i is a sphere, but S^*_i is not. Moreover, suppose that S_{i^c} is not a sphere, either. Then there is a j such that S_j is not a sphere and φ^c induces an embedding of G^*_j in $S^*_{j^c}$ with $S^*_{j^c} = S^*_{i^c}$, by (*i*) (the second part). As shown above, we have $S^*_j \cong S^*_{j^c}$. Since S^*_i is not a sphere, there is a self-singular point in S^*_i. Since $z_1 \neq z_2$ in G, there are at least two vertices, say $u, v \in V_i$ that have been identified into a single vertex in G^*_i. However, φ^c induces an embedding of H''_i in S_{i^c} that arises from the embedding of H_i in S_i (see the note below (*ii*)). Thus, at least one of the vertices u and v, say u, is separated from each vertex from V_j (and also from $V_i - u$) by a cycle in S_{i^c} in φ^c. Since G^*_j is embedded in $S^*_{i^c}$ in φ^c, we have $S^*_{i^c} \succeq S^*_j$, as shown above. Since u is separated from each vertex from $V_j \cup (V_i - u)$ by a cycle in S_{i^c} in φ^c, we have $S^*_{i^c} \succ S^*_j$, which contradicts $S^*_{i^c} \cong S^*_j$. Hence, if S^*_i is not a sphere but S_i is, then S_{i^c} is a sphere, too.

Now analogously as above, if S^*_i is not a sphere but S_i is, we have $S^*_{i^c} \succeq S^*_i$, by (*ii*) and (*iii*). Moreover, if $S^*_{i_k}$ is not a sphere but S_{i_k} is, $1 \leq k \leq 2$, we have $S^*_{i^c_1} \neq S^*_{i^c_2}$. Hence, we have $S^*_{i^c} \cong S^*_i$, since the set of those j for which $S^*_j \succ S^*_i$ is finite. Thus:

(*iv*) *For each i, $1 \leq i \leq l$, φ^c induces an embedding of G^*_i in $S^*_{i^c}$. If S^*_i is not a sphere, we have $S^*_{i^c} \cong S^*_i$. Moreover, if $i_1 \neq i_2$, and $S^*_{i_1}$ and $S^*_{i_2}$ are not spheres, then $S^*_{i^c_1} \neq S^*_{i^c_2}$.*

Necessary conditions for S

To obtain the necessary conditions in Theorem 1, we now need to utilize the "finer structure" of G, that is, the way how G depends on the labelling of the surfaces and the singular points. In fact, we only need to consider the vertices z_1 and z_2 in G.

Suppose that z_1 and z_2 are placed in two self-singular points of S^*_1 in φ. Let z_j be obtained by the identification of t_j vertices of G_1, $1 \leq j \leq 2$. Suppose that G^*_1 is embedded in $S^*_{1^c}$ in φ^c. Since $t_1 + t_2 - 1 > \max\{t_1, t_2\}$, we have $S^*_{1^c} \succ S^*_1$, by (*i*) and (*iii*). By (*iv*) we have:

(1) *No pseudosurface S^*_i contains more than one self-singular point, $1 \leq i \leq l$.*

Suppose that B_S contains a cycle of length at least four. Let $p_1, S_1, p_2, \ldots, p_t, S_t, p_1$ be a shortest cycle in B_S such that $t \geq 2$. Let z_1 be placed in p_1 and z_2 be placed in p_2 in φ. Let $\overline{G}$ be the subgraph of G/z_1z_2 induced by $G^*_2, G^*_3, \ldots, G^*_t$.

By (iv) each G_i^*, $2 \le i \le t$, is embedded in one pseudosurface $S_{i^c}^*$ in φ^c. Hence, $\overline{G}$ is embedded in one pseudosurface, say S_k^*, in φ^c if $t = 2$. Now suppose that $t \ge 3$. Then $G_2^*, \ldots, G_t^*$ are joined to a $(t-1)$-cycle, by (ii). (More precisely, if we replace each G_i^* by a single vertex g_i, and join g_i with g_j by an edge whenever G_i^* and G_j^* have some common vertices, then $\overline{G}$ will result to a cycle on $t-1$ vertices.) Since $2t$ is the length of a shortest cycle of length at least four in B_S, the graph $\overline{G}$ is embedded in one pseudosurface, say S_k^*, if $t \ge 3$.

Since $\overline{G}$ is not a planar graph, S_k^* is not a sphere. By (iv) (the second part), at most one pseudosurface from $S_2^*, \ldots, S_t^*$ is not a sphere. If $S_k^* \not\cong S_i^*$ for each i, $2 \le i \le t$, then the finiteness of the set of those j for which $S_j^* \cong S_k^*$ contradicts (iv). Hence, $S_k^* \cong S_j^*$ for some j, $2 \le j \le t$, and S_j^* is the unique pseudosurface from $S_2^*, \ldots, S_t^*$ which is not a sphere.

Suppose that $t = 2$. Then $\overline{G} = G_j^* = G_2^*$ and the vertex $z_1 z_2$ is embedded in a self-singular point of S_k^* in φ^c. Hence we have $S_k^* \succ S_2^*$, by (iii), which contradicts $S_k^* \cong S_2^*$.

Now suppose that $t \ge 3$. Let $\overline{G}'$ be the subgraph of $\overline{G}$ induced by G_i^*, $2 \le i \le t$ and $i \ne j$. Then $\overline{G}'$ is a connected graph containing two distinct vertices, say z^1 and z^2, of G_j^*. Let z^i be obtained by the identification of a set V_j^i vertices from V_j, and possibly some vertices outside V_j, $1 \le i \le 2$. Since $S_k \cong S_j$, the graph H_j'' is uniquely and faithfully embeddable in S_k. Hence, each pair of vertices from V_j is separated by at least two vertex-disjoint cycles in S_k in φ^c (see the note before (iii)). Hence, also the sets of vertices V_j^1 and V_j^2 are separated by two vertex-disjoint collections of cycles in S_k in φ^c. Since $\overline{G}'$ joins z^1 with z^2 in S_k^* in φ^c, there is a self-singular point of S_k^* that allows this connection. Hence, we have $S_k^* \succ S_j^*$ by (iii), which contradicts $S_k^* \cong S_j^*$. Thus:

(2) *There is no cycle of length greater than two in B_S.*

Thus, S has a "tree structure". Suppose that at least two pseudosurfaces from $S_1^*, \ldots, S_l^*$ are not spheres. Let $S_2, p_1, S_1, p_2, S_3, p_3, \ldots, p_{t-1}, S_t$ be a longest path in B_S such that both S_2^* and S_t^* are not spheres. Suppose that $t \ge 3$. Let z_1 be placed in p_1 and z_2 be placed in p_2 in φ.

Let $\overline{G}$ be the subgraph of $G/z_1 z_2$ induced by $G_2^*, G_3^*, \ldots, G_t^*$. By (iv) each G_i^*, $2 \le i \le t$, is embedded in one pseudosurface $S_{i^c}^*$ in φ^c. Moreover, $G_2^*, \ldots, G_t^*$ are joined to a $(t-1)$-path, by (ii). (More precisely, if we replace each G_i^* by a single vertex g_i, and join g_i with g_j by an edge whenever G_i^* and G_j^* have some common vertices, then $\overline{G}$ will result to a path on $t-1$ vertices.) Thus, there are surfaces, say S_{i_1} and S_{i_2}, at distance $2t$ in B_S, such that $S_{i_1}^*$ and $S_{i_2}^*$ are not spheres, and either $S_{i_1}^*$ or $S_{i_2}^*$ are covered by planar graphs, possibly empty, in φ^c, which contradicts (iv). Hence:

(3) *If S_i^* and S_j^* are not spheres, then S_i and S_j are at distance two in*

B_S.

Let S_1^* and S_2^* be not spheres. Let pS_1 and pS_2 be edges of B_S, and let z_1 be placed in p in φ. Suppose that S_1^* contains a self-singular point different from p that is occupied by z_2 in φ. Since p is the unique singular point lying in at least two pseudosurfaces that are not spheres, by (2) and (3), the vertex z_1z_2 is placed in p in φ^c, by (*ii*).

Let p be a self-singular point of S_i^*, $2 \leq i \leq l$. Then p is a self-singular point of $S_{i^c}^*$, by (*ii*) and (3). However, p is a self-singular point of $S_{1^c}^*$, while p is not a self-singular point of S_1^*, by (1). Since there is just a finite set of S_j^* that contains p as a self-singular point, by (*iv*) (the second part) we have:

(4) *If S_i^* and S_j^* are not spheres and p is a self-singular point of S_i^*, then $p \in S_j^*$.*

Hence, if there are three pseudosurfaces, say S_1^*, S_2^*, and S_3^*, that are not spheres, then they are glued in a unique singular point p, by (2) and (3). Moreover, if one of them, say S_1^*, contains a self-singular point p', then $p' = p$, by (4). Thus, the spherically-irreducible subspace of S contains at most one singular point. This completes the proof. $\square$

We remark that if a pseudosurface S is not minor-closed, then there are infinitely many graphs G embeddable in S such that G/xy is not embeddable in S for some $xy \in E(G)$, by Lemma 3. The problem of determining whether or not the embeddability in a given non-minor-closed pseudosurface can be characterized by a finite set of forbidden subgraphs remains open.

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