TWO-RADIALLY MAXIMAL GRAPHS WITH SPECIAL CENTERS

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Abstract. A graph G is two-radially maximal if G is noncomplete and for each pair (u, v) of its nodes with distance two the addition of the new edge uv to G decreases its radius. We prove that the central subgraph of any two-radially maximal graph contains an edge, and we show that those of them that have a star as the central subgraph are sequential joins of complete graphs.

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1. Introduction

The concept of radius of a graph is frequently used in graph theory. It reflects properties that are interesting in applications and it also plays an important role in the theory. In this paper graphs having extremal properties with respect to radius are studied.

The terminology is based on [1]. Our graphs are undirected, without loops and multiple edges, but they may have an infinite number of nodes. Let G be a graph. By $V(G)$ is denoted the node set of G. By the distance $d_G(u, v)$ or $d(u, v)$ of the nodes u and v we mean the length of a shortest path joining u and v (the $(u{-}v)$-geodetic). The *eccentricity* $e_G(v)$ or $e(v)$ of a node v equals to $\max\{d(v, u) : u \in V(G)\}$, and the radius $r(G)$ equals to $\min\{e(v) : v \in V(G)\}$. The nodes with the minimum eccentricity are called *central* and they induce the *center* $C(G)$ of the graph G.

A survey on centers can be found in Buckley - Harary [1]. It is known that the center of a tree is either K_1 or K_2. Further, there are only seven graphs admissible as centers for maximal outerplanar graphs [7]. Centers of chordal graphs are studied in [6] and those of line graphs in [4] and [5]. In [5] it is shown that any connected i-iterated line graph is a center of some i-iterated line graph for $i \in \{0, 1, 2\}$, which generalizes a result of Buckley, Miller, and Slater [2]. Here we deal with centers of graphs that possess some properties related to their radius. Moreover, we give a condition that secures that the center of a graph contains an edge.

A graph G is radially maximal if G is noncomplete and the addition of any new edge to G decreases its radius. In [3] it is shown that any graph can be an induced subgraph of some regular radially maximal graph with a prescribed radius $r \geq 3$, and hence, the class of these graphs is rich in a sense. Here we show that the result is different if we search for radially maximal graphs with a prescribed center. Note that the cycle C_{2k} is radially maximal for $k \geq 2$, and $C(C_{2k}) = C_{2k}$. Hence, the class of centers of radially maximal graphs is infinite.

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Here we slightly weaken the notion of radially maximal graphs. A graph G is two-radially maximal if G is noncomplete and for each pair (u,v) of its nodes such that $d(u,v)=2$ we have $r(G+uv)<r(G)$. Clearly, each radially maximal graph is two-radially maximal. On the other hand, each noncomplete path on even number of nodes is two-radially maximal but not radially maximal. We will prove that the center of any two-radially maximal graph contains an edge. Moreover, we present a class of graphs that are not the center of any two-radially maximal graph.

By the sequential join $G_1+G_2+\cdots+G_n$ of graphs $G_1,G_2,\ldots,G_n$ we mean a graph G such that $V(G)=V(G_1)\cup V(G_2)\cup\cdots\cup V(G_n)$, and two nodes $x\in V(G_i)$ and $y\in V(G_j)$ are adjacent, if either $i=j$ and xy is an edge of G_i or $|i-j|=1$ (see [1, pp. 26]). Let K^i be a complete graph, $1\le i\le 2t$ and $t\ge 0$. Then for each $n\ge 2$ and α, $1\le\alpha\le n-1$, the sequential join

$$K_1+K^1+K^2+\cdots+K^t+K_\alpha+K_{n-\alpha}+K^{t+1}+K^{t+2}+\cdots+K^{2t}+K_1,$$

is a two-radially maximal graph with center K_n. Main Theorem deals with other two-radially maximal graphs with centers of radius one. We remark that if $W\subseteq V(G)$, then $\langle W\rangle$ denotes the subgraph of G induced by W. A node s is called *universal* if it is adjacent to all other nodes, i.e. $e(s)=1$.

Main Theorem. *Let G be a graph.*
1. If u and v are nodes of G such that $d(u,v)=2$ and $r(G+uv)<r(G)$, then $C(G)$ contains an edge.
2. Let a graph H contain a universal node, and let G be two-radially maximal with center consisting of H and possibly some isolated nodes. Then $C(G)=H$, and for some subset $W\subseteq V(H)$ the graphs $\langle W\rangle$ and $\langle V(H)-W\rangle$ are complete, and each node from $V(H)-W$ is adjacent to some node from W.
3. If G is two-radially maximal and $C(G)$ is a star (i.e. a complete bipartite graph $K_{1,s}$, $s\ge 1$), then $C(G)=K_2$ and

$$G\cong K_1+K^1+\cdots+K^t+K_1+K_1+K^{t+1}+\cdots+K^{2t}+K_1\,, \qquad (1)$$

where K^i are complete graphs for $i=1,2,\ldots,2t$ and $t=r(G)-2\ge 0$.

If one replace "$d(u,v)=2$" with "$d(u,v)=k$", $k\ge 3$, in Part 1 then the result does not hold. (Let G be a path on $2k+3$ nodes. Let u be a neighbor of the central node, and let v be the node of degree two such that $d(u,v)=k$. Then $r(G+uv)=k<k+1=r(G)$, but the center of G contains no edges.) Joining the two nodes with the greatest distance in some graph of type (1) we obtain a new graph of the same radius. Hence, the original graph is not radially maximal and the following Corollary holds.

Corollary. *A center of any radially maximal graph contains at least two edges.*

The graph H_n in Fig. 1 is radially maximal with radius $n+5$, and the center of H_n contains three edges, namely the thick ones. (By P_n is denoted a path on n nodes, $n\ge 1$.) But the following problem is still open.

Problem. *Are there radially maximal graphs whose centers contain just two edges?*

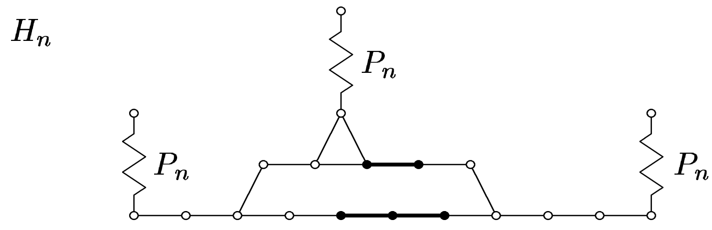

Fig. 1

Further, it can be interesting to characterize graphs that are centers of some radially maximal graph (two-radially maximal graph).

2. Proofs

A node y is *eccentric* to c if $d(c,y)=e(c)$. First we introduce a certain distance related concept. Let c be a node in a connected graph G. By $nut(c)$ we mean the set of nodes x in G such that for every node y eccentric to c we have

$$e(c)=d(c,y)=d(c,x)+d(x,y) \tag{2}$$

(Hence, x lies on some $(c-y)$-geodetic for every node y eccentric to c). Note that if a node z lies in $nut(c)$, then any node that lies on some $(c-z)$-geodetic lies in $nut(c)$, and hence the graph $\langle nut(c)\rangle$ is connected.

Lemma 1. *Let x and c be two adjacent nodes in a connected graph G, c be a central node and $x\in nut(c)$. Then x is a central node, too.*

Proof. For each node y eccentric to c we have $d(x,y)=e(c)-d(c,x)=e(c)-1$ by (2), since $x\in nut(c)$. Further, for any node z not eccentric to c we have $d(x,z)\le d(x,c)+d(c,z)\le 1+e(c)-1=e(c)$. Hence $e(x)\le e(c)$ and x is a central node. □

Lemma 2. *Let G be a connected graph, and let $u,v\in V(G)$ such that $d(u,v)=2$. Then $r(G+uv)<r(G)$ if and only if there is a central node c such that c is not an isolated node in $C(G)$, $\{u,v\}\in nut(c)$, and $|d(u,c)-d(v,c)|=2$.*

Proof. First suppose that for the nodes u and v in G with $d(u,v)=2$ there is a central node c such that $\{u,v\}\in nut(c)$ and $|d(u,c)-d(v,c)|=2$. Then $r(G+uv)\le e_{G+uv}(c)=e_G(c)-1=r(G)-1$, and hence $r(G+uv)<r(G)$.

Now assume $r(G+uv)<r(G)$ for the nodes u and v with $d(u,v)=2$. Then we have $r(G+uv)=r(G)-1$. Since $e_{G+uv}(z)\in\{e_G(z)-1,e_G(z)\}$ for any $z\in V(G)$, $C(G+uv)$ is a subgraph of $C(G)$. Let $c\in V(C(G+uv))\subseteq V(C(G))$. Since $e_{G+uv}(c)<e_G(c)$, we have $2\le |d_G(c,v)-d_G(c,u)|\le d(u,v)=2$, say $d_G(c,v)-d_G(c,u)=2$. Let y be any node eccentric to c in G, and let P be a $(c-y)$-geodetic in $G+uv$. Clearly, P contains the edge uv, since $d_{G+uv}(c,y)\le e_{G+uv}(c)<e_G(c)=d_G(c,y)$. Let $x\in V(G)$ such that $xu,xv\in E(G)$. Now if the edge uv in P is replaced by the path uxv, we obtain a $(u-v)$-geodetic in G that contains u and v. Hence $\{u,v\}\in nut(c)$.

Finally, since $nut(c)\ne\{c\}$ and $nut(c)$ induces a connected subgraph, there is $y\in nut(c)$ that is adjacent to c in G. By Lemma 1 $y\in V(C(G))$, and hence c is not an isolated node in $C(G)$. □

Lemma 3. *Let s be a universal node in a graph H, and let G be a two-radially maximal graph with center consisting of H and possibly some isolated nodes. Then for any two nodes $u, v \in V(G){-}V(H)$ we have*

$$d(u,v) = 2 \quad \Rightarrow \quad |d(s,u) - d(s,v)| = 2. \tag{3}$$

Further, for any node w with $d(s,w) = 2$, the nodes of H adjacent to w induce a complete graph.

Proof. Let $u, v \in V(G){-}V(H)$ and $d(u,v) = 2$. By Lemma 2 there is a node $c \in V(C(G))$ nonisolated in $C(G)$, hence $c \in V(H)$, such that $|d(c,u) - d(c,v)| = 2$, say

$$d(c,u) - d(c,v) = 2. \tag{4}$$

If $c = s$ then (3) holds. Let $c \neq s$. If s lies on some $(c{-}v)$-geodetic, and hence on some $(c{-}u)$-geodetic, we have $d(c,v) = d(c,s){+}d(s,v)$ and $d(c,u) = d(c,s){+}d(s,u)$. Now (4) implies (3).

We have $u, v \notin V(H)$, and hence $c \neq v$. Assume s does not lie on any $(c{-}v)$-geodetic, and let $P = ca \ldots v$ (maybe $a = v$) be some $(c{-}v)$-geodetic. Since $v \in nut(c)$ we have $a \in nut(c)$, and Lemma 1 yields $a \in V(C(G))$. Hence $a \in V(H)$ and s is adjacent to both c and a. Now we prove $d(s,v) = d(c,v)$ and $d(s,u) = d(c,u)$, which will imply (3). We prove only $d(s,v) = d(c,v)$, since the proof of the second equality is very similar.

We have $d(s,v) \geq d(c,v)$ since $d(c,v) < d(c,s){+}d(s,v) = 1{+}d(s,v)$. On the other hand, $d(s,v) \leq d(s,a){+}d(a,v) = 1{+}d(c,v){-}1 = d(c,v)$, and hence $d(s,v) = d(c,v)$. Hence (3) is proved.

Now we prove that for any node w with $d(s,w) = 2$, the nodes adjacent to w in H induce a complete graph. Assume to the contrary that there are two nonadjacent nodes $p, q \in V(H)$, both adjacent to w. Then $d(p,q) = 2$, and Lemma 2 yields that there is a central node $c \in V(H)$ such that $|d(c,p) - d(c,q)| = 2$ and $p, q \in nut(c)$. Since $\max\{d(c,p), d(c,q)\} \leq d(H) \leq 2$, we have $c \in \{p,q\}$, say $c = p$. (We remark that by $d(H)$ is denoted the diameter of H.) Then $q \in nut(p)$, and since w lies on a $(p{-}q)$-geodetic $w \in nut(p)$. By Lemma 1 we have $w \in V(C(G))$, which contradicts $d(s,w) = 2$. $\square$

Proof of Main Theorem. Part 1 is implied by Lemma 2.

Part 2. Let G be two-radially maximal, and let $C(G)$ consist of a graph H (with a universal node s) and possibly some isolated nodes. Denote $N_i(G) = \{w \in V(G){-}V(H) : d(s,w) = i\}$, $i \geq 0$. The graphs $\langle N_i(G) \cup N_{i+1}(G)\rangle$ are unions of complete graphs, since they do not contain nodes with distance two, by (3). Now we construct a graph F, whose nodes are those of H together with cliques of $\langle N_i(G)\rangle$, $i \geq 1$. Two nodes of F are adjacent if and only if the corresponding nodes or cliques of G are adjacent by an edge. Using (3) one can verify that if two cliques, say F_1 and F_2, are adjacent, then for every $x \in V(F_1)$ and $z \in V(F_2)$, the x and z are adjacent in G. Since G is noncomplete, we have $C(F) = C(G)$. Moreover, F is two-radially maximal, since joining two nodes with distance two in F by an edge corresponds to joining at least two nodes with distance two in G.

Further, note that $\langle N_i(F) \cup N_{i+1}(F)\rangle$ $(i \geq 0)$ is a collection of isolated edges and nodes. Hence $\langle V(F){-}V(H)\rangle$ is the union of paths. We remark that $N_1(F)$ contains at most one node, by (3).

Let y be a node eccentric to s and let $saw\dots y$ be an $(s-y)$-geodetic. Let W be the set of nodes from $V(H)$ adjacent to w. Then $\langle W\rangle$ is complete by Lemma 3. In what follows we prove that $\langle V(H)-W\rangle$ is complete and each node $g\in V(C(G))-W$ is adjacent to some node of W. The w lies on each $(g-y)$-geodetic, since $\langle V(F)-V(H)\rangle$ is the union of paths. Thus, $d(g,y)=d(g,w)+d(w,y)=d(g,w)+r(G)-2\leq r(G)$, and since $d(g,w)\geq 2$, we have $d(g,w)=2$. Let t be the common neighbor of g and w. Then either $t\in W$ or $t\in N_1(F)$, according to the structure of F and hence G. We show that we can assume that $t\in W$.

Suppose that $t\in N_1(F)$, and distinguish two cases:

Case 1. Let $w\in nut(s)$. Then stw is a $(s-w)$-geodetic, and hence $t\in nut(s)$. Thus $t\in C(F)=C(G)$ by Lemma 1, a contradiction.

Case 2. Let $w\notin nut(s)$. Then there exists a node $y'\neq y$, such that $d(s,y')=r(G)$ and w does not lie on any $(s-y')$-geodetic. Let w' be the node of $N_2(F)$ which lies on some $(s-y')$-geodetic. Then $w'\neq w$ and w and w' are nonadjacent (it follows from the definition of F). Note that t and w' are not adjacent, since otherwise $d(w,w')=2$ and (3) yields $|d(s,w)-d(s,w')|=2$, a contradiction. Now we can interchange w with w' and y with y', and we have $t\in W$, since $N_1(F)$ contains at most one node as mentioned above.

Thus, each node $g\in V(C(G))-W$ is adjacent to some node of W, and hence $C(G)=H$.

Finally, we prove that $\langle V(H)-W\rangle$ is complete. Assume to the contrary that there are two nonadjacent nodes u and v in $\langle V(H)-W\rangle$. Then $d(u,v)=2$, and we have $|d(c,u)-d(c,v)|=2$ for some $c\in V(H)$, by Lemma 2. Since $d(H)\leq 2$, we have $c\in\{u,v\}$, say $c=u$. Then $v\in nut(u)$. Since y is eccentric to both u and v, we have $r(F)=d_F(u,y)=d_F(u,v)+d_F(v,y)=2+r(F)$, a contradiction. Hence, the graph $\langle V(H)-W\rangle$ is complete.

Part 3. Let H be a star. Then H contains a universal node, and hence $C(G)=K_2=C(F)$ by Part 2. (In the case $C(G)=K_{1,2}$ we have $s\in V(H)-W$, and hence W consists of one node. Thus, there is a node from $V(H)-W$ not adjacent to some node from W.) It is easy to verify that F is a path, and hence, G is the sequential join of complete graphs. Clearly, the first one and the last one are K_1, since G is two-radially maximal. □

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