CENTERS IN ITERATED LINE GRAPHS

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Abstract. For a graph G such that $L^2(G)$ is not empty, we construct a supergraph H such that $C(L^i(H)) = L^i(G)$ for all i, $0 \le i \le 2$. Here $L^i(G)$ denotes the i-iterated line graph of G and $C(G)$ denotes the subgraph of G induced by central nodes. This result is, in a sense, best possible since we provide an infinite class of graphs G such that $L^i(G) \neq C(L^i(H))$ for any graph $H \supseteq G$ and all $i \geq 3$.

In this paper we study centers in connected iterated line graphs. Since the line graph transformation is very natural and some NP-complete problems are polynomial for line graphs, the class of line graphs is of interest. A survey on centers can be found in [1].

By $d_G(u, v)$ we denote the distance between the nodes u and v in G. Then the eccentricity, $e_G(u)$, of the node u is the maximum $d_G(u, v)$ taken over all the nodes of G. The *center*, $C(G)$, of an arbitrary connected graph G is the subgraph of G induced by its central nodes, i.e. the nodes having the minimal eccentricity. It is known that each graph G can be the center of some graph H, where $|V(H)| \leq |V(G)|+4$ (see [1, p.41]). Moreover, Buckley, Miller and Slater [2] have shown that for each graph G with $n \geq 9$ nodes and an integer $k \geq n+1$ there exist a k-regular graph H having G as a center. By now little is known about centers of special graphs. Clearly, the center of a tree consists of either a single node or a pair of adjacent nodes. All seven central subgraphs admissible in maximal outerplanar graphs were listed by Proskurowski [6]. Laskar and Shier [5] studied centers in chordal graphs. Further it was shown [3] that the class of possible centers of line graphs is very rich, namely, for each graph G without isolated nodes there is a graph H such that $C(H) = G$ and $C(L(H)) = L(G)$. Here if G is a nontrivial graph then by its line graph $L(G)$ we mean such a graph whose nodes are the edges of G and two nodes in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. By i-iterated line graph of G we mean $L^i(G)$, where $L^0(G) = G$ and $L^{i+1}(G) = L(L^i(G))$, for an integer $i \geq 1$.

Here we show that any i-iterated line graph is a center of some i-iterated line graph if $i \leq 2$.

Theorem 1. *Let G be a graph and $L^2(G)$ is not empty. Then there is a graph $H \supseteq G$ such that:*
$L^i(G) = C(L^i(H))$ *for* $i = 0, 1, 2$.
$L^3(G) = C(L^3(H))$ *if G is triangle-free and $L^3(G)$ is not empty.*

Before proving Theorem 1, we recall some notations and results that can be found in [4]. We will identify edges in a graph G with the corresponding nodes in $L(G)$. Hence if u and v are two adjacent nodes in G then by uv we mean an edge in G as well as the node in $L(G)$ corresponding to the edge uv. This notation enables us to consider a node in $L^i(G)$ $(i \geq 2)$ as a pair of adjacent nodes in $L^{i-1}(G)$, either of these is a pair of adjacent nodes from $L^{i-2}(G)$, and so on. Furthermore, we can define each node v in $L^i(G)$ using only edges of G, and such a definition will be called the *recursive definition of v in G.*

If G is a graph and v is a node in $L^i(G)$, $(i \geq 1)$, then by the *k-butt* $B_k(v)$ of the node v in $L^i(G)$ we mean the subgraph of $L^{i-k}(G)$ induced by the edges involved into the recursive definition of the node v. The k-butts are characterized in Lemma 2.

Lemma 2 ([4]). *Let J be a subgraph of $L^{i-k}(G)$ and $L^i(G)$ is not empty $(i \geq 1)$. Then J is a k-butt for some node in $L^i(G)$ if and only if J is a connected graph with at most k edges, that is not isomorphic to a path with less than k edges.*

Further, the distance $d(H, J)$ between two subgraphs H and J of a graph G equals to the length of a shortest path in G joining a node from H to a node from J. The following lemma enables us to compute distances between nodes in iterated line graphs.

Lemma 3 ([4]). *Let G be a connected graph, $L^i(G)$ is not empty for an integer $i \geq 1$, and let u and v be distinct nodes in $L^i(G)$. Then*

(S1) $d(u, v) = i + d(B_i(u), B_i(v))$ *if the i-butts of v and u are edge-disjoint.*
(S2) $d(u, v) = \max\{t; t\text{-butts of } u \text{ and } v \text{ are edge-disjoint}\}$ *if i-butts of u and v have a common edge.*

Now we prove Theorem 1:

Proof of Theorem 1. Let $n = |V(G)|$. We construct supergraph H from G by adding some nodes and edges in three steps.

(1) For each node x of G, we add $2 \cdot n + 14$ nodes and $4 \cdot n + 12$ edges. Ten out of the added nodes we denote by $a_x, b_x, \ldots, h_x, u_x, v_x$ (see Fig. 1).
(2) For each pair x, y of nodes of G, we add $2 \cdot n + 16$ another new nodes and $4 \cdot n + 16$ edges. Ten out of the added nodes we denote by $a_{x,y}, b_{x,y}, \ldots, h_{x,y}, u_{x,y}, v_{x,y}$ (see Fig. 2).
(3) If G does not contain triangles then, for each triple x, y, z of nodes of G, we add $2 \cdot n + 16$ another new nodes and $4 \cdot n + 16$ edges.

Eight out of the added nodes we denote by
$a_{x,y,z}, b_{x,y,z}, \ldots, f_{x,y,z}, u_{x,y,z}, v_{x,y,z}$ (see Fig. 3).

Here u_x and v_x, $u_{x,y}$ and $v_{x,y}$, $u_{x,y,z}$ and $v_{x,y,z}$ are joined to each node of G except for x, x and y, x and y and z, respectively, by edge-disjoint paths of the length two.

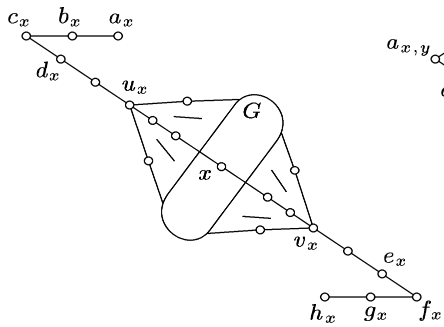

Fig. 1

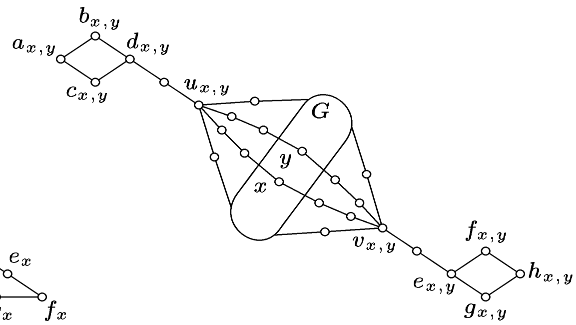

Fig. 2

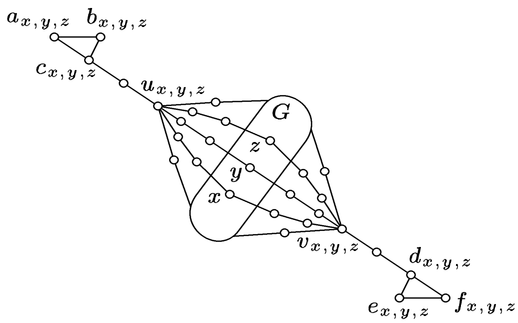

Fig. 3

At first we prove $C(H) = G$. Let $w \in V(G)$. Then $d_H(w, z) \le 5$ for each node $z \in V(G)$ because of the node u_w (see Fig. 1). Moreover, we have $e_H(w) = 8$, since $d_H(w, a_w) = 8$ and $d_H(w, a_{w,y}) = 7$ and $d_H(w, a_{w,y,z}) = 6$ for arbitrary $y, z \in V(G)$.

Let w be a node of H but not the node of G. We distinguish two cases:

(1) There is $x \in V(G)$ such that $wx \in E(H)$.
Then $d_H(w, a_x) = 9$ or $d_H(w, h_x) = 9$, so $e_H(w) \ge 9$.

(2) There is not $x \in V(G)$ such that $wx \in E(H)$.
Now x can be chosen arbitrarily and $d_H(w, a_x) \ge 9$ or $d_H(w, h_x) \ge 9$, so $e_H(w) \ge 9$.

Thus, $C(H) = G$.

Now assume that G does not have triangles. We prove that $C(L^3(H)) = L^3(G)$. We shall investigate the distances between butts of nodes of $L^3(G)$.

Let w be a node of $L^3(G)$. We point out that $|V(B_3(w))| = 4$ according to Lemma 2, since G contains no triangles. Then, $d_H(B_3(w), B_3(z)) \le 4$ for each node z of $L^3(G)$ (see Fig. 1). Thus, we have $d_{L^3(H)}(w, z) \le 7$ by Lemma 3. However, we have $e_{L^3(H)}(w) = 7$, since $d_H(B_3(w), \{a_x, b_x, c_x, d_x\}) = d_H(B_3(w), \{a_{x,y}, b_{x,y}, c_{x,y}, d_{x,y}\}) = d_H(B_3(w), \{a_{x,y,z}, b_{x,y,z}, c_{x,y,z}\}) = 4$ for arbitrary $x, y, z \in V(G)$.

Let w be a node of $L^3(H)$ but not the node of $L^3(G)$. Denote by $S_{x,y,z}$ the node set $\{a_{x,y,z}, b_{x,y,z}, c_{x,y,z}\}$ and by $S'_{x,y,z}$ the node set $\{d_{x,y,z}, e_{x,y,z}, f_{x,y,z}\}$. We distinguish four cases according to the number of nodes in the intersection of $B_3(w)$ and $V(G)$:

(1) $B_3(w) \bigcap V(G) = \{x, y, z\}$.
Then $d_H(B_3(w), S_{x,y,z}) = 5$ or $d_H(B_3(w), S'_{x,y,z}) = 5$ (see Fig. 3) and so $e_{L^3(H)}(w) \geq 8$.

(2) $B_3(w) \bigcap V(G) = \{x, y\}$.
We choose z such that $d_H(B_3(w), \{u_{x,y,z}\}) = 3$ or $d_H(B_3(w), \{v_{x,y,z}\}) = 3$. Such a choice is possible since G has at least four nodes (G is triangle-free and $L^3(G)$ is not empty). Then $d_H(B_3(w), S_{x,y,z}) = 5$ or $d_H(B_3(w), S'_{x,y,z}) = 5$ and so $e_{L^3(H)}(w) \geq 8$.

(3) $B_3(w) \bigcap V(G) = \{x\}$.
We choose y and z such that $d_H(B_3(w), \{u_{x,y,z}\}) = 3$ or $d_H(B_3(w), \{v_{x,y,z}\}) = 3$. Again, such a choice is possible since G has at least four nodes. Then $d_H(B_3(w), S_{x,y,z}) = 5$ or $d_H(B_3(w), S'_{x,y,z}) = 5$ and so $e_{L^3(H)}(w) \geq 8$.

(4) $B_3(w) \bigcap V(G) = \emptyset$.
Then we can choose x, y and z arbitrarily and $d_H(B_3(w), S_{x,y,z}) \geq 5$ or $d_H(B_3(w), S'_{x,y,z}) \geq 5$. So, $e_{L^3(H)}(w) \geq 8$.

Thus, $C(L^3(H)) = L^3(G)$.

The assertions $C(L(H)) = L(G)$ and $C(L^2(H)) = L^2(G)$ can be proved analogously. □

The graph H constructed in the proof above does not have the minimal order since it is not necessary to add nodes to all pairs and triples of nodes of G.

However, the following statement implies that the constraints on i in Theorem 1 are necessary for an arbitrary graph G.

Theorem 4. *Let G be a graph in which each node lies in a triangle and G contains at least two edge-disjoint triangles. Moreover, let $L^i(G)$ do not be a self-centred graph for some $i \geq 3$. Then $L^i(G) \neq C(L^i(H))$ for any graph $H \supseteq G$.*

Proof. Suppose to the contrary that there is H such that $C(L^i(H)) = L^i(G)$. Since $L^i(G)$ is not a self-centred graph, there is an edge $e \in E(H) - E(G)$ with a node, say x, in G and a triangle T in G containing x. From (S1) in Lemma 3 we have $e_{L^i(H)}(z) \geq i$ for each node z of $L^i(G)$ since G contains two edge-disjoint triangles and all the central nodes have the same eccentricity. Let w_1 and w_2 be nodes of $L^i(H)$ such that $B_i(w_1) = T$ and $K_{1,3} \cong B_i(w_2) \subseteq T \cup e$. Then w_1 is a node of $L^i(G)$ and w_2 is not a node of $L^i(G)$. However, for each node z of $L^i(H)$ such that $d_{L^i(H)}(w_1, z) \geq i$ we have $d_{L^i(H)}(w_1, z) \geq d_{L^i(H)}(w_2, z)$ from Lemma 3. Thus, $e_{L^i(H)}(w_1) \geq e_{L^i(H)}(w_2)$, since $e_{L^i(H)}(w_1) \geq i$. The proof is complete since we have arrived to a contradiction. □

Note that the square of a path on at least five nodes satisfies the hypothesis of Theorem 4 for all $i \geq 3$. The square of a graph G is the graph whose nodes correspond to those of G, and where two distinct nodes are joined whenever the distance between them is at most two. On the other hand, one can see that the sufficient condition in Theorem 4 is not necessary. (Identify a node of a triangle with an endnode of a path on at least four nodes and take this graph as G). Thus, the characterization of the graphs G satisfying $C(L^i(H)) = L^i(G)$ for all i, $0 \leq i \leq k$, and some supergraph H, remains an unsolved problem (here $k \geq 3$).

Since the center of a graph is its induced subgraph, G is a center of some line graph if and only if G is a line graph. However, the center of i-iterated line graph is not necessarily an i-iterated line graph. (Let H be the graph obtained from $K_{1,4}$ by inserting two nodes into one edge. Then the center of $L^2(H)$ is isomorphic to K_4, that is not a 2-iterated line graph.) Thus, we have the following problem:

Problem. *For $i \geq 2$, characterize the centers of i-iterated line graphs*

References

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