NOTE ON LINEAR ARBORICITY

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Abstract. The conjecture of linear arboricity requires to decompose any n-regular graph into $\lceil \frac{n+1}{2} \rceil$ linear forests. Here a new approach to this conjecture is developed. We bound the degrees in forests by $\lfloor \frac{n+1}{2} \rfloor$.

Introduction

In this note, a graph will always mean a finite undirected graph without loops and multiple edges. A graph Γ is n-regular if the degree of each vertex in Γ is n. We emphasize that the letter n will always be used only in this meaning.

A letter T will indicate a forest. A linear forest is a forest with all vertex degrees less or equal to 2. For any graph Γ, the arboricity $\Upsilon(\Gamma)$ of Γ (the linear arboricity $\Xi(\Gamma)$ of Γ), is the minimum number of edge disjoint forests (linear forests), whose union is Γ.

Symbols $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of a graph Γ, respectively. An edge joining two vertices x and y we denote as xy.

The degree of vertex x in a graph Γ (a forest T) is denoted as $deg_\Gamma(x)$ ($deg_T(x)$). The greatest degree in a graph Γ is denoted as $\Delta(\Gamma)$.

For a real number v, $\lfloor v \rfloor$ denotes the lower integer part of v and $\lceil v \rceil = -\lfloor -v \rfloor$.

In 1961 C. St. J. A. Nash-Williams [9] and W. T. Tutte [12] have determined the arboricity of arbitrary graph. In particular

$$\Upsilon(\Gamma) = \left\lceil \frac{n+1}{2} \right\rceil$$

for an n-regular graph Γ.

The following conjecture on linear arboricity is due to J. Akiyama, G. Exoo and F. Harrary [3].

Conjecture 1. *For an arbitrary n-regular graph Γ,*

$$\Xi(\Gamma) = \left\lceil \frac{n+1}{2} \right\rceil \quad .$$

The inequality $\Xi(\Gamma) \geq \left\lceil \frac{n+1}{2} \right\rceil$ follows from $\Xi(\Gamma) \geq \Upsilon(\Gamma)$. The converse is not known. However, the conjecture has been proved in some special cases.

For n=3, 4 , it was proved by J. Akiyama, G. Exoo and F. Harrary in [3] and [4]. For n=5, 6, 8 , it was proved by H. Enomoto and B. Peroche in [6], for n=6 , by P. Tomasta in [11] and for n=10 by F. Guldan in [7].

In general, as we mentioned above, the linear arboricity is at least $\left\lceil \frac{n+1}{2} \right\rceil$. Already in 1981 it was shown in [4] that $\Xi(\Gamma) \leq \left\lceil \frac{3}{2} \left\lceil \frac{n}{2} \right\rceil \right\rceil$ for any n-regular graph Γ. In 1987 N. Alon [5] proved by probabilistic methods, that for arbitrary $\epsilon > 0$ and n sufficiently large the linear arboricity of an n-regular graph is less than $\left(\frac{1}{2} + \epsilon\right) \cdot n$.

The problem of linear arboricity in multigraphs was studied by H. Aït-djafer [1], [2].

In this note we attempt to look at the problem from another point of view. As we mentioned above, we have $\Upsilon(\Gamma) = \left\lceil \frac{n+1}{2} \right\rceil$ for an arbitrary n-regular graph Γ. Let $\Delta_n[\mathcal{R}]$ denotes the maximum degree of vertices over all components in decomposition $\mathcal{R}$ of an n-regular graph to $\left\lceil \frac{n+1}{2} \right\rceil$ forests. Hence, $\Delta_n[\mathcal{R}] \leq n$ is the best possible inequality which can be derived from [9] and [12], because the authors admit vertices of arbitrary degree. However, Conjecture 1 requires to find a decomposition $\mathcal{R}$, satisfying $\Delta_n[\mathcal{R}] = 2$.

Up to date, no better bounds are known in general. In this note we show that $\Delta_n[\mathcal{R}] \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. A short proof of Conjecture 1 for $n = 3$ using techniques similar to those used in the proof of Theorem 1 can be found in [8].

Main results

All proofs here are constructive. We decompose a graph Γ into forests T_i , $i = 1, 2, \ldots, h$.

We use elementary operation of inserting i-admissible edge xy into forest T_i , $i = 1, 2, \ldots, h$. Let k be a constant to which we decrease the value of $\Delta_n[\mathcal{R}]$. An edge $xy \notin E(T_i)$ is i-admissible iff:

(i) $T_i \cup xy$ is a forest
(ii) $deg_{(T_i \cup xy)}(x) \leq k$
(iii) $deg_{(T_i \cup xy)}(y) \leq k$

We note that inserting an i-admissible edge into a forest T_i cannot increase the number of vertices of degree greater then k in forests T_j , $j = 1, 2, \ldots, h$.

We set $h = \left\lceil \frac{n+1}{2} \right\rceil$. The following identity will often be used.

$$n - h + 1 = n + 1 - \left\lceil \frac{n+1}{2} \right\rceil = n + 1 + \left\lfloor \frac{-n-1}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$$

Theorem 1. *Let Γ be an n-regular graph, $n > 3$. Then there are $h = \left\lceil \frac{n+1}{2} \right\rceil$ edge disjoint forests $T_1, T_2, \ldots, T_h$ covering Γ such that $\Delta(T_i) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, $i = 1, 2, \ldots, h$.*

Proof. Assume that there is a graph Γ which cannot be decomposed into forests, where $\Delta(T_i) \leq \lfloor \frac{n+1}{2} \rfloor$, $i = 1, 2, \ldots, h$.

By C. St. J. A. Nash-Williams [9] and [10], there is a decomposition of Γ into h forests. We can assume that the decomposition is chosen so that the number of vertices $z \in V(\Gamma)$ with $deg_{T_i}(z) > \lfloor \frac{n+1}{2} \rfloor$ for any i, is minimum.

Let x be a vertex with $deg_{T_i}(x) > \lfloor \frac{n+1}{2} \rfloor = n - h + 1$. Without loss of generality let $i = 1$. In the following we modify our decomposition of Γ to a new one with $deg_{T_1}(x) = n - k + 2$, and then we determine the degrees of some vertices in T_i.

Since $2(n-h+2) \geq n+2$, the only forest T_i with $deg_{T_i}(x) > \lfloor \frac{n+1}{2} \rfloor$ is T_1. Let $deg_{T_i}(x) = \lfloor \frac{n+1}{2} \rfloor + j$. Since $n-(n-h+1+j) = h-j-1$, there are j forests say $T_2, T_3, \ldots, T_{j+1}$ with $deg_{T_i}(x) = 0$ for all $i \in \{2, 3, \ldots, j+1\}$. Since $n - h + 2 \geq 2$ if $n > 3$, there are at least two vertices y with $xy \in E(T_1)$. Let y be such that $xy \in E(T_1)$. Since $1+2(n-h+1) \geq n+1$, we have $deg_{T_i}(y) < n-h+1$ for some $i \in \{2, 3, \ldots, j+1\}$ if $j \geq 2$. Assume $deg_{T_{j+1}}(y) < n - h + 1$. Then xy is $(j+1)$-admissible and we can insert xy into T_{j+1}. We decreased $deg_{T_1}(x)$ by one.

Now we have $j - 1$ forests $T_2, T_3, \ldots, T_j$ with $deg_{T_i}(x) = 0$ for all $i \in \{2, 3, \ldots, j\}$. Let y be such that $xy \in E(T_1)$. If $j - 1 \geq 2$, xy is i-admissible for some $i \in \{2, 3, \ldots, j\}$ and we can insert xy into T_i.

Thus, $j - 1$ neighbours of x in T_1 we can insert into T_i, $i \in \{2, 3, \ldots, j+1\}$. Then $deg_{T_1}(x) = n - h + 2$ and there is a forest say T_2 with $deg_{T_2}(x) = 0$. But $deg_{T_2}(y) \geq n - h + 1$ for all y with $xy \in E(T_1)$ since otherwise we get a contradiction with the original choice of $T_1, T_2, \ldots, T_k$ in Γ.

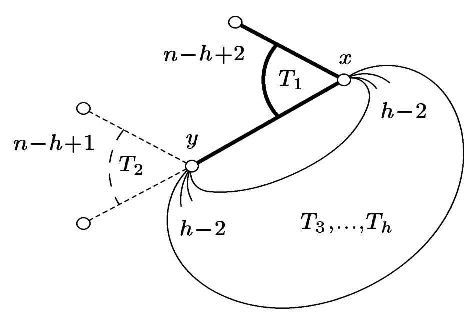

Fig. 1

The edge xy is i-admissible if x and y are in distinct components of T_i, $i > 2$, because of $(n - h + 2) + (n - h + 1) \geq n + 1$ (see Fig. 1). Thus,

$$deg_{T_1}(x) = n - h + 2 \quad \text{and} \quad deg_{T_i}(x) = 1,\ i = 3, 4, \ldots, h,$$
$$deg_{T_2}(y) = n - h + 1 \quad \text{and} \quad deg_{T_i}(y) = 1,\ i = 1, 3, 4, \ldots, h$$

for all y with $xy \in E(T_1)$ because Γ is n-regular (see Fig. 1).

Let y be a fixed vertex of $V\Gamma$ with $xy \in E(T_1)$. Then x and y are joined by a path in T_3. Let us denote $y{=}a_0, a_1, \ldots, a_m{=}x$ the vertices of this path (see Fig. 2).

We claim $deg_{T_1}(a_1) \geq n - h + 1$. Otherwise we can insert $a_1 a_0$ into T_1 and xy into T_3. We get again forests because xy is 3-admissible if $a_1 a_0 \notin E(T_3)$, and $a_1 a_0$ is 1-admissible if $xy \notin E(T_1)$. But then

$deg_{T_1}(x) = n - h + 1$ that is a contradiction with the original choice of $T_1, T_2, \ldots, T_k$ in Γ.

Vertices a_0 and a_1 must be in the same component of T_i, $i = 4, 5, \ldots, h$. Otherwise we can insert a_1a_0 into T_i and xy into T_3, because a_1a_0 is i-admissible. Thus, $deg_{T_i}(a_1) \geq 1$, $i > 3$.

We have the following identities:

$deg_{T_3}(a_1) = 2, \quad deg_{T_1}(a_1) = n - h + 1, \quad deg_{T_i}(a_1) = 1,\ i > 3,$
because $n = deg_\Gamma(a_1) \geq 2 + (n - h + 1) + h - 3 = n$.

Analogously, we show that $deg_{T_2}(a_2) \geq n - h + 1$ because otherwise we can insert a_1a_2 into T_2 and xy into T_3. Similarly a_2 and a_1 must be in the same component of T_i for each $i > 3$ because otherwise we can insert a_1a_2 into T_i and xy into T_3 (see Fig. 2). It means that:

$deg_{T_3}(a_2) = 2, \quad deg_{T_2}(a_2) = n - h + 1, \quad deg_{T_i}(a_2) = 1,\ i > 3.$

We can repeat this construction till a_m is reached. Finally we obtain:

$deg_{T_3}(x) = 1, \quad deg_{T_1}(x) = n - h + 2, \quad deg_{T_i}(x) = 1,\ i > 3.$

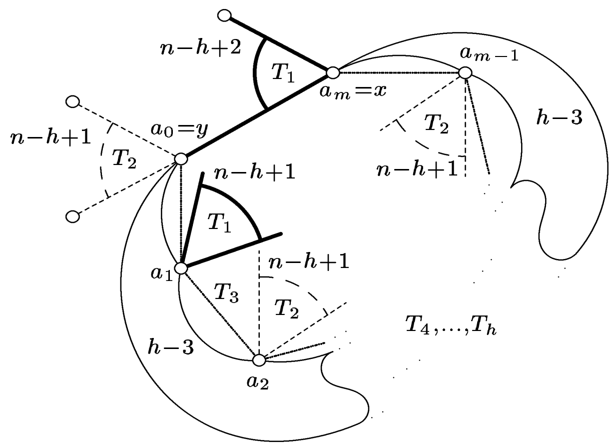

Fig. 2

Hence $deg_{T_3}(x){=}deg_{T_3}(y){=}1$ and $deg_{T_3}(a_i){=}2$, $i = 1, 2, \ldots, m{-}1$. But as we mentioned above, there exists $\overline{y} \neq y$ with $\overline{y}x \in E(T_1)$. We obtain existence of a path in T_3 with vertices $\overline{y} = b_0, b_1, \ldots, b_{\overline{m}} = x$ by an analogous process. Here $deg_{T_3}(\overline{y}) = deg_{T_3}(x) = 1$ and $deg_{T_3}(b_i) = 2$, $i = 1, 2, \ldots, \overline{m} - 1$. It means that x, y and $\overline{y}$ are three distinct vertices of degree 1 in linear tree that is a contradiction.

This concludes the proof of Theorem 1. $\square$

We have proved that every n-regular graph, for which $n > 3$, can be decomposed into $\lceil \frac{n+1}{2} \rceil$ forests with maximum degree $\lfloor \frac{n+1}{2} \rfloor$. Assumption $n > 3$ was used to establish three forests, which yield the path $a_0, a_1, \ldots, a_m$ in the proof. Since $\lfloor \frac{n+1}{2} \rfloor = 2$ if $n = 4$, we proved the Conjecture 1 for $n = 4$.

Every graph of degree not greater than k can be completed to k-regular graph by adding new vertices and edges. So Theorem 1 imply that each graph Γ with $\Delta(\Gamma) = k$ can be decomposed into $\lceil \frac{k+1}{2} \rceil$ forests of degree not greater than $\lfloor \frac{k+1}{2} \rfloor$. The decreasing of degrees in forests to some function asymptotically equal even to $o(n)$ is still open.

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