1. Zistite, či rad $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)$ konverguje a ak áno, nájdite jeho súčet.

$$\frac{1}{2^n}+\frac{1}{3^n}=\frac{3^n+2^n}{6^n}<\frac{5^n}{6^n}=\left(\frac{5}{6}\right)^n$$

$a^n+b^n<(a+b)^n$ pre $a,b>0$, $n\geq 2$

$\frac{5}{6}<1 \Rightarrow \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n$ konverguje

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right) << \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n$$

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)}=\sum_{n=1}^{\infty}\frac{1}{2^n}+\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}+\frac{1}{3}\,\frac{1}{1-\frac{1}{3}}=*$$

$a_1=\frac{1}{2}$, $q_1=\frac{1}{2}$ $\qquad b_1=\frac{1}{3}$, $q_2=\frac{1}{3}$

$$*=\frac{1}{2-1}+\frac{1}{3-1}=1+\frac{1}{2}=\boxed{\frac{3}{2}}$$

2. Zistite, či rad $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ konverguje a ak áno, nájdite jeho súčet.

$$\frac{1}{n(n+1)} = \underbrace{\frac{1}{n^2+n} < \frac{1}{n^2}} \implies \underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{\text{konverguje}} \ll \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{konverguje}}$$

konverguje $\Leftarrow$ konverguje

Súčet: rozklad na parciálne zlomky

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)} \qquad A, B \in \mathbb{R}$$

$$1 = A(n+1) + Bn$$

$$1 = n(A+B) + A$$

$$\boxed{\begin{aligned} A+B &= 0 \\ A &= 1 \end{aligned}} \quad \boxed{B = -1}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \dots \boxed{= 1}$$

3. Zistite, či rad $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ konverguje a ak áno, nájdite jeho súčet.

$\ln n < n$ pre $n \geq 1$

$\ln n < \sqrt{n} \cdot \sqrt{n} \quad / \cdot \frac{1}{\sqrt{n}} \cdot \frac{1}{\ln n}$

$\frac{1}{\sqrt{n}} < \frac{\sqrt{n}}{\ln n}$

$$\sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} < \boxed{\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n} \quad \text{diverguje}}$$

diverguje $\Rightarrow$

4. Zistite, či rad $\displaystyle\sum_{n=2}^{\infty} \frac{1000^n}{n!}$ konverguje.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^n}{n!}} = \lim_{n\to\infty} \frac{1000^{\cancel{n+1}} \cdot \cancel{n!}}{\cancel{1000^n} \cdot (n+1)\cancel{!}} =$$

$$= \lim_{n\to\infty} \frac{1000}{\underbrace{n+1}_{\to\infty}} = 0 < 1 \implies \text{Daný rad konverguje.}$$

5. Zistite, či rad $\displaystyle\sum_{n=2}^{\infty} \frac{3^n n!}{n^n}$ konverguje.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}(n+1)!}{(n+1)^{n+1}}}{\dfrac{3^n n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{3^{n+1}}\cancel{(n+1)!}\, n^n}{\cancel{3^n}\cdot \cancel{n!}\,(n+1)^{n+1}} =$$

$$= 3 \lim_{n\to\infty} \frac{\cancel{(n+1)}\overbrace{n\, n \ldots n}^{n}}{\underbrace{\cancel{(n+1)}\underbrace{(n+1)\ldots(n+1)}_{n}}_{n+1}} = 3\lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n = 3\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1-1} =$$

$$= 3\lim_{n\to\infty} \underbrace{\left(1-\frac{1}{n+1}\right)^{n+1}}_{e^{-1}} \cdot \underbrace{\underbrace{\left(1-\frac{1}{n+1}\right)}_{\to 1}{}^{-1}}_{\to 1} = \frac{3}{e} > 1 \Rightarrow \text{Daný rad diverguje.}$$

6. Zistite, či rad

$$\frac{1000}{1} + \frac{1000.1001}{1.3} + \frac{1000.1001.1002}{1.3.5} + \frac{1000.1001.1002.1003}{1.3.5.7} + \cdots$$

konverguje.

$$\sum_{n=1}^{\infty} \frac{\frac{(999+n)!}{999!}}{1.3.\ldots.(1+2(n-1))}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(\overbrace{999+n+1}^{1000+n})!}{\cancel{999!}}}{\frac{\cancel{1.3.5.\ldots}(1+2n)}{\frac{(999+n)!}{\cancel{999!}}}} \quad \text{(n+1) nepárnych čísel} \quad =$$

$$\underbrace{\cancel{1.3.5.\ldots.(1+2(n-1))}}_{n \text{ nepárnych čísel}}$$

$$= \lim_{n\to\infty} \frac{(1000+n)\cancel{!}}{\cancel{(999+n)!}.(1+2n)} = \lim_{n\to\infty} \frac{1000+n}{1+2n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1000}{n}+1}{\frac{1}{n}+2} =$$

$$\frac{1000}{n} \to 0, \quad \frac{1}{n} \to 0$$

$$= \frac{1}{2} < 1 \implies \text{Daný rad } \underline{\text{konverguje.}}$$

7. Zistite, či rad $\sum_{n=2}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}$ konverguje.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{\frac{\cancel{n}(n-1)}{\cancel{n}}} =$$

$$= \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{n-1} = \lim_{n\to\infty} \left(\frac{n+1-2}{n+1}\right)^{n-1} = \lim_{n\to\infty} \left(1-\frac{2}{n+1}\right)^{n+1-2} =$$

$$= \lim_{n\to\infty} \underbrace{\left(1-\frac{2}{n+1}\right)^{n+1}}_{e^{-2}} \cdot \underbrace{\left(\underbrace{1-\frac{2}{n+1}}_{\to 1}\right)^{-2}}_{\to 1} = \frac{1}{e^2} < 1$$

Daný rad konverguje.

8. Zistite, či rad $\sum_{n=2}^{\infty} \frac{2+(-1)^n}{2^n}$ konverguje.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{2+(-1)^n}}{2} = \frac{1}{2} < 1 \Rightarrow$$

$\sqrt[n]{2+(-1)^n} \to 1$

Daný rad konverguje.

$$2+(-1)^n = \left\langle \begin{matrix} 1 \\ 3 \end{matrix} \right\} \Rightarrow \sqrt[n]{2+(-1)^n} = \sqrt[n]{\{1,3\}}$$

$$\lim_{n\to\infty} \sqrt[n]{c} = 1$$

$1 \leq c$

9. Zistite, či rad $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$ konverguje.

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}} =$$

$$= \lim_{n\to\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}\to 0} = (\bar{c})^2 \underline{\underline{< 1}}$$

$$\boxed{\lim_{n\to\infty}\frac{\ln n}{n} = 0} \qquad 0 \le \underbrace{\left(\frac{1+\cos n}{2+\cos n}\right)}_{\bar{c}<1} \le \frac{2}{3}$$

Podľa Cauchyho kritéria daný rad <u>konverguje</u>.