1. Vypočítajte limitu postupnosti $\lim_{n\to\infty} \frac{\cos(2n^3-3n^2+n-5)}{n^2-1}$.

ohraničená funkcia

$$-1 \leq \cos(2n^3-3n^2+n-5) \leq 1$$

$$\lim_{n\to\infty} \frac{\cos(2n^3-3n^2+n-5)}{n^2-1} = \underline{\underline{0}}$$

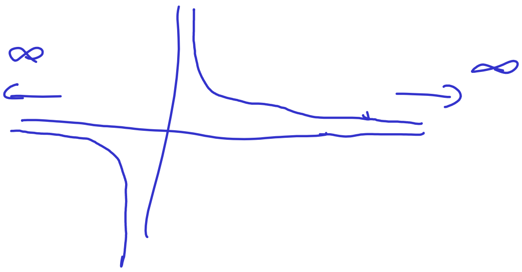
$n^2-1 \to \infty$

2. Vypočítajte limitu postupnosti $\lim\limits_{n\to\infty} \frac{(-1)^n}{n}$.

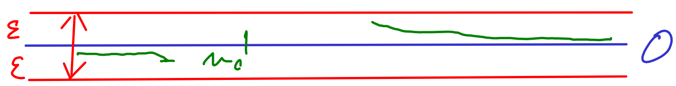
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \underline{\underline{0}}$$

$$\frac{(-1)^n}{n} = \begin{cases} \frac{1}{n} & \text{pre } n \text{ párne} \\ -\frac{1}{n} & \text{pre } n \text{ nepárne} \end{cases} \qquad \lim_{n\to\infty} \frac{1}{n} = 0$$

$f(x) = \frac{1}{x}$



$$\forall \varepsilon > 0 \ \exists n_0 : \forall n > n_0 \ \left| \frac{(-1)^n}{n} \right| < \varepsilon$$



3. Vypočítajte limitu postupnosti $\lim\limits_{n\to\infty} \dfrac{1+2+\cdots+n}{n^2+2n+5}$.

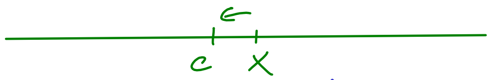
$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2+2n+5} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n^2+n}{n^2+2n+5}\cdot\underbrace{\frac{\frac{1}{n^2}}{\frac{1}{n^2}}}_{=1} =$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{1+\overbrace{\frac{1}{n}}^{\to 0}}{1+\underbrace{\frac{2}{n}+\frac{5}{n^2}}_{\to 0}} = \frac{1}{2}$$

4. Vypočítajte limitu postupnosti $\lim\limits_{n\to\infty} \left\lfloor \sqrt[3]{n^3+3n^2} - n \right\rfloor$

Poznámka: $\lfloor x \rfloor$ označuje dolnú celú časť čísla x.



$\boxed{\lfloor 3{,}14 \rfloor = 3}$ $\quad \lfloor -3{,}14 \rfloor = -4$

$\boxed{a^3 - b^3 = (a-b)(a^2+ab+b^2)}$

$$\lim_{n\to\infty} \left\lfloor \Big(\underbrace{\sqrt[3]{n^3+3n^2}}_{a} - \underbrace{n}_{b}\Big) \cdot \frac{\sqrt[3]{(n^3+3n^2)^2} + n\sqrt[3]{n^3+3n^2} + n^2}{\underbrace{\sqrt[3]{\ldots} + n\sqrt[3]{\ldots} + n^2}_{=1}} \right\rfloor =$$

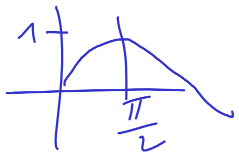
$$= \lim_{n\to\infty} \left\lfloor \frac{\cancel{n^3} + 3n^2 - \cancel{n^3}}{\sqrt[3]{(n^3+3n^2)^2} + \sqrt[3]{n^6+3n^5} + n^2} \cdot \underbrace{\frac{\frac{1}{n^2}}{\frac{1}{n^2}}}_{=1} \right\rfloor =$$

$$= \lim_{n\to\infty} \left\lfloor \underbrace{\frac{3}{\underbrace{\sqrt[3]{1+\underbrace{\ldots}_{\to 0}} + \sqrt[3]{1+\underbrace{\tfrac{3}{n}}_{\to 0}} + 1}_{\to 3^+}}}_{\to 1^-} \right\rfloor \qquad \frac{3}{3^+} \qquad \underline{\underline{= 0}}$$

5. Dokážte, že rad $\sum_{n=1}^{\infty} \arcsin\left(\frac{n^2+n+1}{n^2+1}\right)$ diverguje.

Nutná podmienka konvergencie: $\sum_{n=1}^{\infty} a_n$ $\quad \boxed{\lim_{n\to\infty} a_n = 0}$

$$\lim_{n\to\infty} \arcsin\Bigg(\underbrace{\frac{n^2+n+1}{n^2+1}}_{\to 1}\Bigg) = \arcsin 1 = \frac{\pi}{2} = \frac{3{,}14...}{2} \neq 0$$



$\Downarrow$

Daný rad

DIVERGUJE

6. Zistite, či geometrický rad $\sum_{n=1}^{\infty} (\operatorname{arctg} 1)^n$ konverguje a ak áno, nájdite jeho súčet.

$$\underbrace{\operatorname{arctg} 1}_{a_1} + (\operatorname{arctg} 1)^2 + (\operatorname{arctg} 1)^3 + \dots$$

$q = \operatorname{arctg} 1 \implies$ geometrická postupnosť

$$|q| < 1 \implies \exists \sum_{n=1}^{\infty} \underbrace{a_1 q^{n-1}}_{a_n} = \boxed{\frac{a_1}{1-q}}$$

$$\operatorname{arctg} 1 = \frac{\pi}{4} = \boxed{\frac{3{,}14\ldots}{4} < 1} \qquad \operatorname{tg} \frac{\pi}{4} = 1$$

$\implies q < 1$

$$\sum_{n=1}^{\infty} (\operatorname{arctg} 1)^n = \frac{\frac{\pi}{4}}{1-\frac{\pi}{4}} = \frac{\pi}{4} \cdot \frac{1}{1-\frac{\pi}{4}} = \boxed{\frac{\pi}{4-\pi}}$$

7. Dokážte, že rad $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverguje.

Majorantné kritérium

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$

$\forall n \in \mathbb{N}: \quad \underbrace{\sqrt{n} \leq n}$

$\Downarrow$

$\frac{1}{n} \leq \frac{1}{\sqrt{n}}$

$\underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3}}_{\text{harmonická postupnosť}} + \dots \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n} \to$ diverguje

$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Div}} \ll \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\text{Div} \Rightarrow \text{Div}$

$a_n = \dfrac{2^n \cdot n!}{n^n}$ 8. Zistite, či rad $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ konverguje.

D'Alambertovo (podielové) kritérium: $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$

$$\lim_{n\to\infty} \frac{\dfrac{2^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}}{\dfrac{2^n \cdot n!}{n^n}} = \lim_{n\to\infty} \frac{2^{\cancel{n+1}\,1}\,(n+1)\cancel{!}\; n^n}{\cancel{2^n}\;\cancel{n!}\;(n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{2.\cancel{(n+1)}\,\overbrace{n\cdot n\cdot \ldots \cdot n}^{n}}{\cancel{(n+1)}\underbrace{(n+1)\ldots\ldots(n+1)}_{n}} = 2\lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n = 2\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1-1} =$$

$$= 2\lim_{n\to\infty} \underbrace{\left(1-\frac{1}{n+1}\right)^{n+1}}_{=e^{-1}} \cdot \left(1-\frac{1}{n+1}\right)^{-1} = \frac{2}{e} < 1$$

$\searrow 1$

$e \doteq 2{,}71$

Daný rad KONVERGUJE

9. Zistite, či rad $\sum_{n=1}^{\infty} \frac{n^8}{2^n+3^n}$ konverguje.

Cauchyho (odmocninové) kritérium: $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$

$$a_n = \frac{n^8}{2^n+3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^8}{2^n+3^n}} = \lim_{n\to\infty} \frac{n^{\frac{8}{n}} \to 1}{\sqrt[n]{2^n+3^n}} = \frac{1}{1\le x\le 5} < 1$$

$$\lim_{n\to\infty} n^{\frac{8}{n}} = \lim_{n\to\infty} e^{\ln n^{\frac{8}{n}}} = \lim_{n\to\infty} e^{\frac{8}{n}\ln n} = e^{8 \lim_{n\to\infty} \frac{\ln n}{n} \to 0} = e^0 = 1$$

8. $\lim_{x\to\infty} \frac{\log_a x}{x} = 0$

$$1 \le \sqrt[n]{2^n+3^n} \le \sqrt[n]{5^n} = 5$$

$\boxed{1 \le 2^n + 3^n \le 5^n}$ $\forall n \in \mathbb{N}$

$a^n + b^n \le (a+b)^n$

$a, b > 0$

Daný rad KONVERGUJE