Vyšetrovanie priebehu funkcie $f(x)$ zahŕňa:

1. Nájdenie definičného oboru $D(f)$.
2. Nájdenie nulových bodov funkcie f.
3. Nájdenie intervalov monotónnosti funkcie f.
4. Nájdenie lokálnych extrémov funkcie f.
5. Nájdenie intervalov, na ktorých je funkcia f konvexná/konkávna.
6. Nájdenie asymptot bez smernice aj so smernicou ku grafu funkcie f.
7. Načrtnutie grafu funkcie f. → v bodoch nespojitosti

Niekedy sa explicitne požaduje aj určenie oboru hodnôt $H(f)$ a nájdenie inflexných bodov funkcie f. Oboje samozrejme musíme poznať ak chceme načrtnúť graf funkcie f, rovnako ako potrebujeme poznať aj hodnoty funkcie f v jej kritických bodoch ako sú extrémy, prienik s osou o_y a inflexné body.

1. Vyšetrite priebeh funkcie $f(x)$ a načrtnite jej graf.

$$f(x) = x + \frac{2x}{x^2-1} = \frac{x^3+x}{x^2-1}$$

- $D(f) = \mathbb{R} - \{\pm 1\}$ $\qquad x^2-1 \neq 0 \iff x^2 \neq 1 \iff x \neq \pm 1$

- nulové body: $f(x) = 0 \qquad \frac{x^3+x}{x^2-1} = 0 \iff x^3 + x = 0 \iff x(\underbrace{x^2+1}_{\neq 0}) = 0$

$$\boxed{x = 0}$$

- body nespojitosti: $\{\pm 1\}$

- intervaly monotónnosti: $f'(x) = 1 + \frac{2(x^2-1) - 2x \cdot 2x}{(x^2-1)^2} = \dots = \frac{x^4 - 4x^2 - 1}{(x^2-1)^2}$

stacionárne body: $f'(x) = 0$ $\qquad x^4 - 4x^2 - 1 = 0$, $t = x^2$ $\qquad t^2 - 4t - 1 = 0$

$t_{1,2} = 2 \pm \sqrt{5}$

$t \geq 0$ $\qquad \cancel{2-\sqrt{5}} < 0$

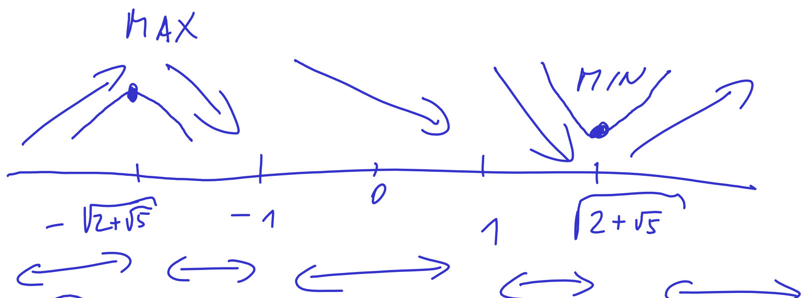
$x^2 = 2 + \sqrt{5}$

$x_1 = -\sqrt{2+\sqrt{5}}$ $\qquad y_1 \doteq f(x_1) \doteq -3{,}33$

$x_2 = \sqrt{2+\sqrt{5}} \doteq 2{,}05$ $\qquad y_2 \doteq f(x_2) \doteq 3{,}33$

rastie: $(-\infty, -\sqrt{2+\sqrt{5}})$, $(\sqrt{2+\sqrt{5}}, \infty)$

klesá: $(-\sqrt{2+\sqrt{5}}, -1)$, $(-1, 1)$, $(1, \sqrt{2+\sqrt{5}})$



$f'(x)$: $+ \quad - \quad - \quad - \quad +$

- lokálne extrémy: $\quad$ lok. max: $[-\sqrt{2+\sqrt{5}}, y_1]$ $\quad X_{max}$

  lok. min: $[\sqrt{2+\sqrt{5}}, y_2]$ $\quad X_{min}$

- konvexnosť / konkávnosť $\qquad f'(x) = \frac{x^4 - 4x^2 - 1}{(x^2-1)^2}$

$$f''(x) = \frac{(4x^3 - 8x)(x^2-1)^{\cancel{2}\,1} - (x^4 - 4x^2 - 1)\,2 \cdot \cancel{(x^2-1)} \cdot 2x}{(x^2-1)^{\cancel{4}\,3}} = \dots$$

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3} \qquad f''(x)=0 \Leftrightarrow 4x\underbrace{(x^2+3)}_{>0} = 0$$

$$x = 0$$

	∩		∪		∩	∪
		-1		0		1
$f''(x)$:	$-$		$+$		$-$	$+$

Konvexná: $(-1,0)$, $(1,\infty)$

Konkávna: $(-\infty,-1)$, $(0,1)$

- Asymptoty bez smernice

$\{\pm 1\}$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{\overbrace{x^3+x}^{\to -2}}{\underbrace{x^2-1}_{\to 0^+}} = \infty \qquad \lim_{x\to -1^+} \frac{\overbrace{x^3+x}^{\to -2}}{\underbrace{x^2-1}_{\to 0^-}} = -\infty$$

$$\lim_{x\to 1^-} \frac{\overbrace{x^3+x}^{\to 2}}{\underbrace{x^2-1}_{\to 0^-}} = -\infty \qquad \lim_{x\to 1^+} \frac{\overbrace{x^3+x}^{\to 2}}{\underbrace{x^2-1}_{\to 0^+}} = \infty$$

$$\boxed{x=-1} \quad \boxed{x=1}$$

- Asymptoty so smernicou

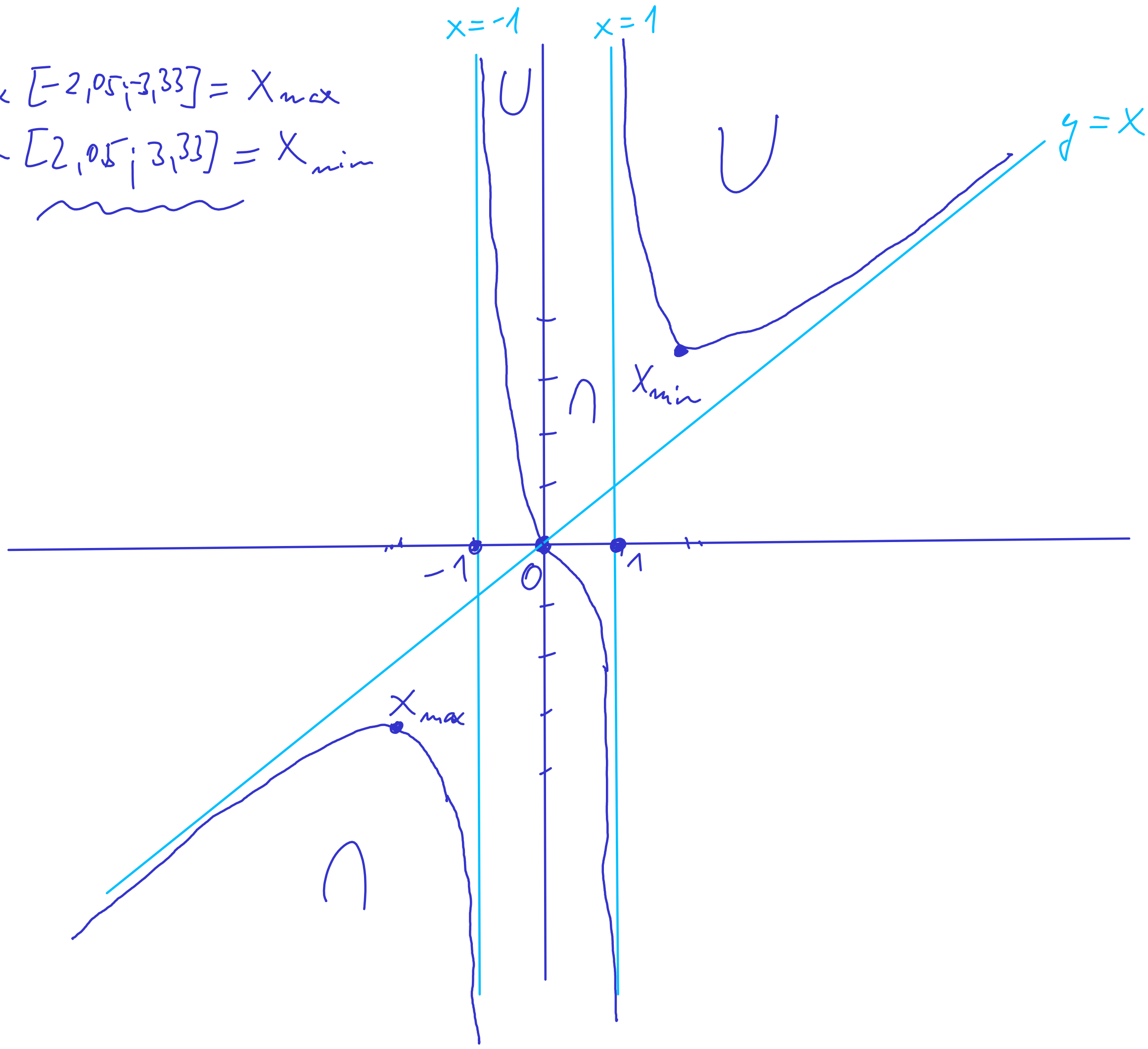
$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{x^3+x}{x^2-1}}{x} = \lim_{x\to\pm\infty} \frac{x^3+x}{x^3-x} = \underline{\underline{1}}$$

$$q = \lim_{x\to\pm\infty} (f(x)-kx) = \lim_{x\to\pm\infty} \left(\frac{x^3+1}{x^2-1} - x\right) = \lim_{x\to\pm\infty} \frac{2x}{x^2-1} = \underline{\underline{0}}$$

$$\boxed{y = x}$$

max $[-2{,}05; -3{,}33] = X_{max}$

min $[2{,}05; 3{,}33] = X_{min}$



2.Vyšetrite priebeh funkcie $f(x)$ a načrtnite jej graf.

$$f(x) = x.e^{-\frac{x^2}{2}}$$

• $D(f) = R$ • nulové body: $f(x)=0$ $\quad x \cdot \underbrace{e^{-\frac{x^2}{2}}}_{>0} = 0 \iff \boxed{x=0}$

• body nespojitosti: nemá $\Rightarrow \nexists$ ABS

• intervaly monotónnosti

$$f'(x) = 1.e^{-\frac{x^2}{2}} + x\, e^{-\frac{x^2}{2}}(-x) = e^{-\frac{x^2}{2}}(1-x^2)$$

stacionárne body $\quad f'(x)=0 \iff \underbrace{e^{-\frac{x^2}{2}}}_{>0}\underbrace{(1-x^2)}=0$

$x_1=-1 \qquad x_2=1$

$y_1 = -\frac{1}{\sqrt{e}} \qquad y_2 = \frac{1}{\sqrt{e}} \doteq 0,6$

MIN MAX

$-1 \qquad 1$

$f'(x)$: $-$ $+$ $-$

• lokálne extrémy: $\boxed{X_{max} = \left[1, \frac{1}{\sqrt{e}}\right], \; X_{min} = \left[-1, -\frac{1}{\sqrt{e}}\right]}$

• konvexnosť/konkávnosť

$$f''(x) = e^{-\frac{x^2}{2}}(-x)(1-x^2) + e^{-\frac{x^2}{2}}(-2x) = e^{-\frac{x^2}{2}}.x(x^2-3)$$

$$f''(x)=0 \iff e^{-\frac{x^2}{2}}.x\underbrace{(x^2-3)}=0$$

$\boxed{\begin{array}{l} x_3=0 \\ y_3=0 \end{array}}$ $\quad \begin{array}{l} x_4=-\sqrt{3} \\ y_4=-\sqrt{3}\,e^{-\frac{3}{2}} \end{array}$ $\quad \boxed{\begin{array}{l} x_5=\sqrt{3} \\ y_5=\sqrt{3}\,e^{\frac{3}{2}} \end{array}} \doteq 0,39$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
	$\cap$	$\cup$	$\cap$	$\cup$
$f''(x)$:	$-$	$+$	$-$	$+$

- Asymptoty bez smernice

NEMÁ

- Asymptoty so smernicou

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x e^{-\frac{x^2}{2}}}{x} = \lim_{x\to\pm\infty} e^{\overbrace{-\frac{x^2}{2}}^{\to -\infty}} = \underline{\underline{0}}$$

$$q = \lim_{x\to\pm\infty} (f(x) - kx) = \lim_{x\to\pm\infty} \overbrace{x}^{\pm\infty} e^{\overbrace{-\frac{x^2}{2}}^{\to 0}}$$

$\infty \cdot \infty$

$$\lim_{x\to\infty} \underbrace{e^{\ln x}}_{=x} \cdot e^{-\frac{x^2}{2}} = \lim_{x\to\infty} e^{\ln x - \frac{x^2}{2}} = e^{\lim_{x\to\infty} \overbrace{\left(\ln x - \frac{x^2}{2}\right)}^{\to -\infty}} = \underline{\underline{0}}$$

$\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{\left(\ln x - \frac{x^2}{2}\right)}{x} \cdot x = \lim_{x\to\infty} \Bigg(\overbrace{\frac{\ln x}{x}}^{\to 0} - \overbrace{\frac{x}{2}}^{\to \infty} \Bigg) \cdot \overbrace{x}^{\to \infty} = \underline{\underline{\underline{-\infty}}}$$

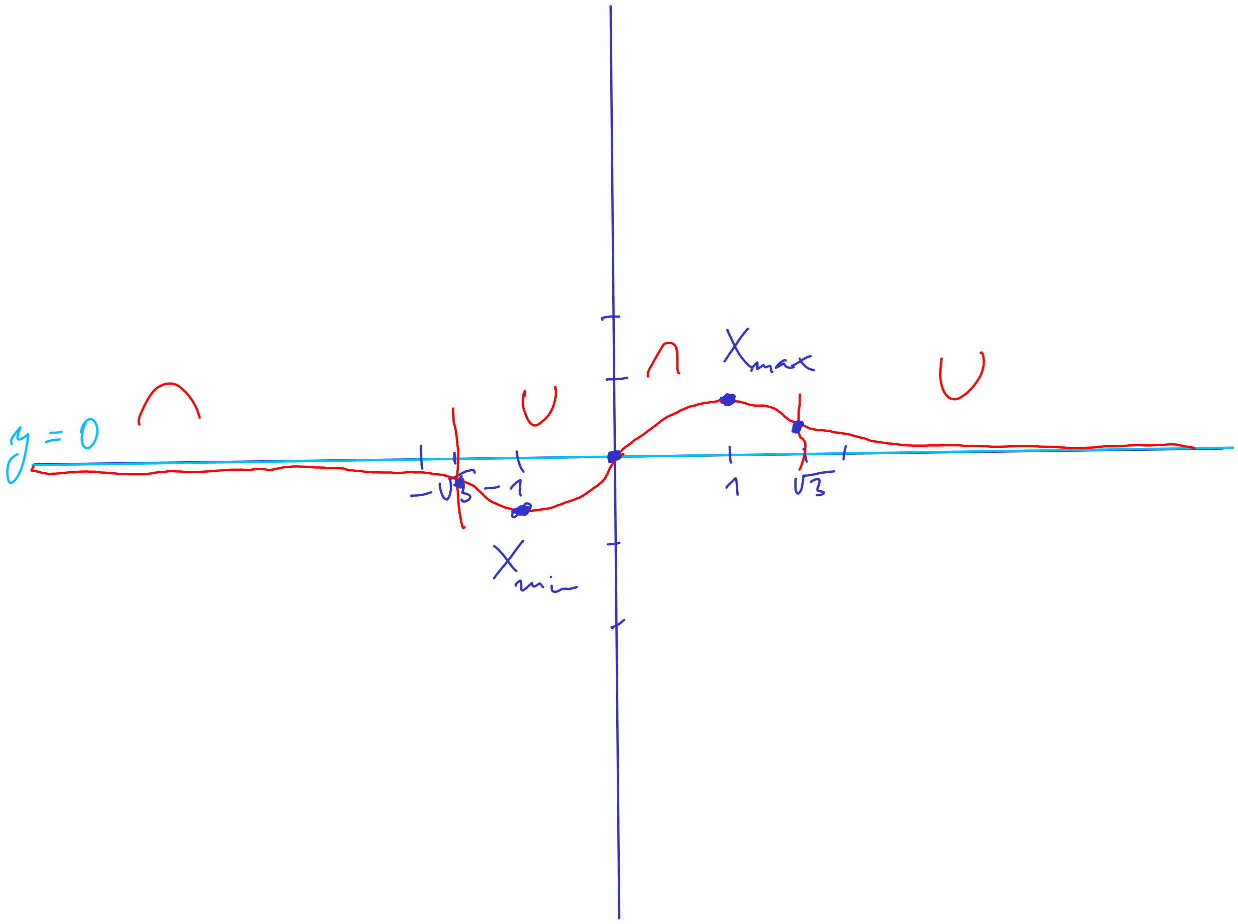
$-\infty \cdot \infty$

pre $x \to \infty$: $q = 0$

$$\lim_{x\to -\infty} \underbrace{-e^{\ln|x|}}_{=x} \cdot e^{-\frac{x^2}{2}} = \lim_{x\to -\infty} -e^{\ln|x| - \frac{x^2}{2}} = \ldots = 0$$

pre $x \to -\infty$: $q = 0$

$$\boxed{y = 0}$$



y = 0
X max
X min
−√3
−1
1
√3

3. Vyšetrite priebeh funkcie $f(x)$ a načrtnite jej graf.

$$f(x) = \ln \frac{1+x}{1-x}$$

- Definičný obor

$$\frac{1+x}{1-x} > 0 \iff (1+x)(1-x) > 0$$

$$1 - x^2 > 0$$

$$-1 < x < 1$$

$$x \in (-1, 1)$$

$$\boxed{D(f) = (-1, 1)}$$

- Body nespojitosti: nemá

- Intervaly monotónnosti:

$$f'(x) = \frac{1-x}{1+x} \cdot \frac{1(1-x) - (1+x)(-1)}{(1-x)^2} = \dots = \frac{2}{1-x^2}$$

stacionárne body: $f'(x) = 0$ $\quad x \in (-1, 1)$

$$\frac{2}{1-x^2} = 0 \qquad \emptyset$$

$f'(x) > 0$ pre $\forall x \in D(f)$ $\quad$ na $D(f)$ $f(x)$ rastie

- Lokálne extrémy: nemá

- Konvexnosť / konkávnosť

$$f'(x) = \frac{2}{1-x^2} = 2(1-x^2)^{-1}$$

$$f''(x) = -2(1-x^2)^{-2} \cdot (-2x) = \frac{4x}{(1-x^2)^2}$$

$$f''(x) = 0 \iff \frac{4x}{(1-x^2)^2} = 0 \iff 4x = 0 \iff \boxed{x = 0}$$

$\cap \quad \cup$

$f''(x)$: $-1 \quad \ominus \quad 0 \quad \oplus \quad 1$

- Asymptoty bez smernice:

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \ln\left(\frac{1+x \to 0^+}{1-x \to 2}\right) = -\infty$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \ln\left(\frac{1+x \to 2}{1-x \to 0^+}\right) = \infty$$

- Asymptoty so smernicou: nemá, pretože $D(f) = (-1, 1)$