On M-valued L-fuzzy bornologies

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Contents

In order to apply the conception of boundedness, so crucial in the theory of metric spaces, as well as in the theory of linear topological spaces to the case of a general topological space Hu Sze-Tsen introduced the notions of a bornology and of a bornological space:
S.-T. Hu, *Boundedness in a topological space*, J. Math. Pures Appl., **78** (1949), 287–320. S.-T. Hu,, *Introduction to General Topology*, Holden-Day, San-Francisko, 1966.

Definition: Bornology and bornological space

Given a set X a bornology on it is a family $\mathcal{B} \subseteq 2^X$ such that

- (1B) $\forall x \in X \quad \Longrightarrow \{x\} \in \mathcal{B}$;

(2B) if $U \subseteq V \subseteq X$ and $V \in \mathcal{B}$, then $U \in \mathcal{B}$;

(3B) if $U, V \subseteq X \quad U, V \in \mathcal{B}$ then $U \cup V \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called a bornological space and the sets from $\mathcal{B}$ are viewed as bounded in this space.

Definition: bounded mappings

Given bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ a mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is called bounded if the image $f(A)$ of every set $A \in \mathcal{B}_X$ belongs to $\mathcal{B}_Y$.

Important examples of bornological spaces $(X, \mathcal{B})$

- a metric space and its bounded subsets (that is sets with finite diameter);
- a topological space and its relatively compact subsets;
- a uniform space and its totally bounded subsets.

Problem of fuzzification of the concept of bornology

Aiming to develop an appropriate concept of bornology in the context of fuzzy sets and fuzzy structures we have to make a choice between different possible ways how it can be done. As a pattern of possible ways for this choice we see the three well developed approaches to extension of the concept of topology to the context of fuzzy sets and fuzzy structures.
Conceptionally generalizing these approaches to the case of a mathematical structure of a sufficiently general nature, we describe them as follows:

Problem of fuzzification of the concept of bornology

(FC) **Fuzzy-Crisp Approach** To consider a crisp analogue of a classical mathematical structure but to use families of fuzzy sets instead of families of ordinary sets.

(CF) **Crisp-Fuzzy Approach** To consider fuzzy analogous of classical mathematical structures in case when the structure itself is fuzzy, but acts on families of crisp sets.

(FF) **Fuzzy-Fuzzy Approach** Consider fuzzy analogues of classical mathematical structures when both the structure itself is fuzzy, and acts on families of fuzzy sets

Lattice L

$L = (L, \leq, \wedge, \vee)$ is a complete lattice, in some cases completely distributive.
$0_L, 1_L$ are respectively the bottom and the top elements of the lattice L.

Lattice M

$M = (M, \leq, \wedge, \vee)$ is a complete lattice, in some cases completely distributive.
$0_M, 1_M$ are respectively the bottom and the top elements of M
$M = (M, \leq, \wedge, \vee, *)$ is a cl-monoid.

Definition: (L, M)-bornologies

An M-valued L-fuzzy bornology on a set X , or just an (L, M)*-bornology* for short, is a mapping $\mathcal{B} : L^X \to M$ such that

(LMB1) $\mathcal{B}(\chi_x) = 1_M$ for each $x \in X$, where

$$\chi_x(y) = \begin{cases} 1_M & \text{if } y = x \\ 0_M & \text{if } y \neq x \end{cases}$$

(LMB2) $A \leq B, A, B \in L^X \Longrightarrow \mathcal{B}(A) \geq \mathcal{B}(B)$;

(LMB3) $A_1, A_2 \in L^X \Longrightarrow \mathcal{B}(A_1 \vee A_2) \geq \mathcal{B}(A_1) * \mathcal{B}(A_2)$.

The pair $(X, \mathcal{B})$ is called *an L-fuzzy bornological space* and L-sets $B \in \mathcal{B}$ are called *bounded* in this space.

Category **BOR**(L, M)

Bounded mapping of M-valued L-fuzzy bornological spaces

Given M-valued L-fuzzy bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ a mapping $f : X \to Y$ is called *bounded* if $\mathcal{B}_Y(f(A)) \geq \mathcal{B}_X(A)$ for every $A \in Ł^X$.

Category **BOR**(L, M)

If $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ and $g : (Y, \mathcal{B}_Y) \to (Z, \mathcal{B}_Z)$ are bounded, then $g \circ f : (X, \mathcal{B}_X) \to (Z, \mathcal{B}_Z)$ is bounded, too. The identity mapping $id_X : (X, \mathcal{B}_X) \to (X, \mathcal{B}_X)$ is bounded. Hence M-valued L-fuzzy bornological spaces and bounded mappings form a category **BOR**(L, M).

Special cases: L-fuzzy bornologies and category $\mathbf{BOR}(L, 2)$

L-fuzzy bornology

An *L-fuzzy bornology* on a set X is a family $\mathcal{B} \subseteq L^X$ such that

(LB1) $\bigvee\{B \mid B \in \mathcal{B}\} = 1_X$;

(LB2) $B \in \mathcal{B}, C \in L^X, C \leq B \Longrightarrow C \in \mathcal{B}$;

(LB3) $B_1, B_2 \in \mathcal{B} \Longrightarrow B_1 \vee B_2 \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called *an L-fuzzy bornological space* and L-sets $B \in \mathcal{B}$ are called *bounded* in this space.

Special cases: L-fuzzy bornologies and category $\mathbf{BOR}(L, 2)$

Bounded mapping of L-fuzzy bornological spaces

Given two L-fuzzy bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ a mapping $f : X \to Y$ is called *bounded* if $f(B) \in \mathcal{B}_Y$ whenever $B \in \mathcal{B}_X$

Category $\mathbf{BOR}(L, 2)$

L-fuzzy bornological spaces and bounded mappings between them form a category $\mathbf{BOR}(L, 2)$ and called *the category of L-fuzzy bornological spaces*.

Special cases: *M*-valued bornologies and category **BOR**$(2, M)$

M-valued bornology

An M-valued bornology on a set X is a mapping $\mathcal{B} : 2^X \to M$ such that

(MB1) $\forall x \in X \quad \Longrightarrow \mathcal{B}(\{x\}) = 1_M$;

(MB2) If $U \subseteq V \subseteq X$ then $\mathcal{B}(V) \leq \mathcal{B}(U)$;

(MB3) $\forall\, U, V \subseteq X$ the inequality $\mathcal{B}(U \cup V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$ holds .

The pair $(X, \mathcal{B})$ is called an L-valued bornological space and the value $\mathcal{B}(A)$ is interpreted as the degree of boundedness of a set A in the space $(X, \mathcal{B})$.

Special cases: *M*-valued bornologies and category $\mathbf{BOR}(2, M)$

Bounded mappings

A mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ where $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ are *L*-valued bornological spaces is called bounded if $\mathcal{B}_X(A) \leq \mathcal{B}_Y(f(A))$ for every $A \in 2^X$.

Category $\mathbf{BOR}(2, M)$

L-valued bornological spaces and bounded mappings form the category of *M*-valued bornological spaces.$\mathbf{BOR}(2, M)$.

M-valued bornologies on powersets of sets

Crisp-Fuzzy Approach

M-valued bornologies on powersets of sets

- A.Šostak and I.Uļjane. Bornological structures in the context of L-fuzzy sets, In: Proceedings of EUSFLAT-13. G.Pasi, J. Montero, D. Ciucci (Eds.), 32, Atlantis Press, 2013, pp. 481-488.
- A.Šostak and I.Uļjane. Bornologies in the context of L-fuzzy sets in: Recent Progress for Topology, Computer Science, Fuzzy Mathematics and Economics, Proc. of WiAT'13, J. Guttierez , T. Kubiak, I. Mardones and M.A. de Prada (Eds.), Bilbao, 2013, pp. 119-130.

Definition

An $(M, *)$-valued bornology on a set X is a mapping $\mathcal{B} : 2^X \to M$ such that

(MB1) $\forall x \in X \quad \Longrightarrow \mathcal{B}(\{x\}) = 1$;

(MB2) If $U \subseteq V \subseteq X$ then $\mathcal{B}(V) \leq \mathcal{B}(U)$;

(MB3) $\forall\ U, V \subseteq X$ the inequality $\mathcal{B}(U \cup V) \geq \mathcal{B}(U) * \mathcal{B}(V)$ holds .

The pair $(X, \mathcal{B})$ is called an $(M, *)$-valued bornological space and the value $\mathcal{B}(A)$ is interpreted as the degree of boundedness of a set A in the space $(X, \mathcal{B})$.

M-valued bornologies

Note that in case $* = \wedge$, the second axiom (MB2) is redundant since it follows from the axiom (MB3) and hence M-valued bornology on a set X can be defined as follows:

Definition

A mapping $\mathcal{B} : 2^X \to M$, where $M = (M, \leq, \wedge, \vee, \wedge))$ is an M-valued bornology if and only if it satisfies the following conditions

(MB1) $\forall x \in X \quad \mathcal{B}(\{x\}) = 1$;

(MB3$'$) $\forall\, U, V \subset X \quad \mathcal{B}(U \cup V) = \mathcal{B}(U) \wedge \mathcal{B}(V)$.

Definition

A mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ where $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ are M-valued bornological spaces is called bounded if $\mathcal{B}_X(A) \leq \mathcal{B}_Y(f(A))$ for every $A \in 2^X$.

M-valued bornological spaces and bounded mappings form the category **BOR**$(2, M)$.

Lattice of $(M, *)$-valued bornologies.

Given a cl-monoid $(M, \leq, \wedge, \vee, *)$ and a set X let $\mathfrak{B}(X, M, *)$ stand for the family of all $(M, *)$-valued bornologies on the set X. We introduce an order relation $\preceq$ on $\mathfrak{B}(X, M, *)$, by

$$\mathcal{B}_1 \preceq \mathcal{B}_2 \iff \mathcal{B}_1(A) \geq \mathcal{B}_2(A)\ \forall A \in 2^X,$$

$(\mathfrak{B}(X, M, *), \preceq)$ is a partially ordered set. Bottom element: $\mathcal{B}_\perp(A) = 1_M$ for all $A \in 2^X$. Top element:

$$\mathcal{B}_\top(A) = \begin{cases} 1_M & \text{if } \mid A \mid < \aleph_0 \\ 0_M & \text{otherwise.} \end{cases}$$

The tuple $(\mathfrak{B}(X, M, *), \preceq, \curlywedge, \curlyvee)$ becomes a complete lattice if the supremum $\curlyvee$ and the infimumum $\curlywedge$ in $(\mathfrak{B}(X, M, *), \preceq)$ are appropriately defined. We define them as follows.

Given a family $\{\mathcal{B}_i : 2^X \to M \mid i \in \mathcal{I}\}$ of $(M, *)$-valued bornologies, we define its supremum

$$\curlyvee_{i\in I}\mathcal{B}_i =: \mathcal{B}^0 : 2^X \to L \text{ by setting } \mathcal{B}^0(A) = \bigwedge_{i\in I} \mathcal{B}_i(A)$$

where $\bigwedge$ is the infimum in the lattice M.

Thus we obtain an $(M, *)$-valued bornology $\curlyvee_{i\in I}\mathcal{B}_i$ on X which is the supremum $\curlyvee$ of the family $\{\mathcal{B}_i : 2^X \to M \mid i \in I\}$ in the partially ordered set $(\mathfrak{B}(X, M, *), \preceq)$.

Construction of an *M*-valued bornology from a family of crisp bornologies

Let K be an approximative subset of M and $\{\mathcal{C}_\alpha \mid \alpha \in K\}$ is a non-increasing family of crisp bornologies on a set X. For a set $A \subseteq X$ we define

$$\mathcal{B}(A) = \lambda \text{ where } \bigvee\{\alpha \in K \mid A \in \mathcal{C}_\alpha\} =: \lambda.$$

Construction of an M-valued bornology from a family of crisp bornologies

Theorem

If the family $\{\mathcal{C}_\alpha \mid \alpha \in K\}$ is lower-semicontinuous:

$$\mathcal{C}_\alpha = \bigcap \{\mathcal{C}_\beta \mid \beta \lhd \alpha, \beta \in K\} \text{ for every } \alpha \in M,$$

then the mapping $\mathcal{B} : 2^X \to M$ is an M-valued bornology. Moreover, $\mathcal{B}_\alpha = \mathcal{C}_\alpha$ for every $\alpha \in K$.

M-valued bornologies induced by fuzzy metrics

M-valued bornologies induced by fuzzy metrics

Fuzzy metrics

Basing on the concept of a statistical metric introduced by K. Menger (Probabilistic geometry, Proc. N.A.S., 37 (1951), 226–229.) and thoroughly investigated by B. Schweizer and A. Sclar(*Statisitcal metric spaces*, Pacific J. Math. **10** (1960) 215–229.), I. Kramosil and J. Michalek introduced the notion of a fuzzy metric (*Fuzzy metrics and statistical metric spaces*, Kybernetika **11** (1975), 336 – 344.) Later A. George and P. Veeramani (*On some results in fuzzy metric spaces*, Fuzzy Sets Syst., **64** (1994) 395–399) slightly modified the original concept of a fuzzy metric. In this work we also base ourselves on George-Veeramani's notion of a fuzzy metric.

Definition

A fuzzy metric on a set X is a pair $(m, \odot)$ such that $m : X \times X \times \mathbb{R}^+ \to [0,1]$ is a fuzzy set, where $\mathbb{R}^+ = (0, +\infty)$, and $\odot$ is a continuous t-norm satisfying the following conditions:

(1GV) $m(x, y, t) > 0 \; \forall \; x, y \in X, \; \forall \; t \in (0, \infty)$;

(2GV) $m(x, y, t) = 1$ if and only if $x = y$;

(3GV) $m(x, y, t) = M(y, x, t) \; \forall \; x, y \in X, \forall \; t \in (0, \infty)$;

(4GV) $m(x, z, t + s) \geq m(x, y, t) \odot m(y, z, s) \forall x, y, z \in X \; \forall t, s \in (0, \infty)$;

(5GV) $m(x, y, -) : \mathbb{R}^+ \to [0, 1]$ is continuous as a function of t for all $x, y \in X$ as a function of t.

The triple $(X, m, \odot)$ is called a fuzzy metric space.

Two types of boundedness in fuzzy metric spaces

Let $(X, m, \odot)$ be a fuzzy metric space, $A \subseteq X$ and $t \in (0, \infty)$

Definition

A set A is locally B-bounded at a level t, if there exist $\varepsilon \in (0, 1)$ and $x_0 \in X$ such that

$$A \subset B_t(x_0, \varepsilon) = \{x \in X \mid m(x_0, x, t) > 1 - \varepsilon\}.$$

A set A is called locally B-bounded if it is locally B-t-bounded for all levels $t \in (0, \infty)$.

Definition

A set A is locally D-bounded at a level t, if $diam_t A > 0$, or, equivalently, if there exists $\varepsilon \in (0,1)$ such that $diam_t A > 1 - \varepsilon$ where the diameter $diam_t A$ of set X at a level t is defined as

$$diam_t A = \inf\{M(x,y,t) \mid x,y \in A\}.$$

A set A is locally D-bounded if it is D-t-bounded at all levels $t \in (0,\infty)$.

Corollary

Let $(m, \odot)$ be a strong fuzzy metric and let $\odot$ have no zero divisors. Then a set A is locally B-t-bounded if and only if it is locally D-t-bounded.

Corollary

If $\odot$ has no zero divisors, then A is locally B-bounded if and only if it is locally D-bounded.

M-valued bornologies induced by fuzzy metrics

Given a fuzzy metric space $(X, m, \odot)$ and $\alpha \in (0,1)$ and φ strictly decreasing continuous bijection $\varphi : (0,\infty) \to (0,1)$. let $\mathcal{C}_\alpha$ be the family of finite unions of $\varphi^{-1}(\alpha)$-bounded subsets. C_α is a crisp bornology on X. Let $\mathcal{B}(A) = \bigvee\{\alpha \in (0,1) \mid A \in \mathcal{C}_\alpha\}$. Thus given a fuzzy metric space $(X, m, \odot)$ we construct an M-valued bornology $\mathcal{B}$. Since $\wedge \geq \odot$ for any t-norm $\odot$, $\mathcal{B}$ is an $(M, \odot)$-bornology also for any t-norm $\odot$.

L-valued bornology on the powerset of a (Chang-Goguen) L-fuzzy topological space

Degree of compactness of subsets of a (Chang-Goguen) L-(fuzzy) topological space

Let $(L, \leq, \wedge, \vee, *)$ be a cl-monpoid. Given a set X, the lattice structure from L is extended L^X and as a result L^X becomes a complete completely distributive lattice. Further let be $\mapsto: L \times L \to L$ is residuation induced by $*$. We define $\hookrightarrow: L^X \times L^X \to L$:
$A \hookrightarrow B = \inf_{x \in X}(A(x) \mapsto B(x))$. Then

$$A_1 \vee A_2 \hookrightarrow B = (A_1 \hookrightarrow B) \wedge (A_2 \hookrightarrow B) \text{ for all } A_1, A_2, B \in L^X$$

We use relation $\hookrightarrow$ to define degree of compactness in an *L*-fuzzy topological space (X, τ) as follows:

$$c(A) = \inf\{\sup A \hookrightarrow \bigvee \mathcal{U}_0 \mid \mathcal{U}_0 \subseteq \mathcal{U}, \mid \mathcal{U}_0 \mid < \aleph_0\} \mid \mathcal{U}_0 \in \mathcal{U}, A \leq \bigvee \mathcal{U}\}$$

Construction of an L-valued bornology on the powerset of a (Chang-Goguen) L-fuzzy topological space

Let the degree of relative compactness for subsets in the space (X, τ) be defined by $rc(A) = \sup\{c(B) \mid A \subseteq B,\ A, B \in 2^X\}$.

Theorem

1. $\forall x \in X \quad rc(\{x\}) = 1$;
2. If $A \subseteq B \subseteq X$ then $rc(B) \leq rc(A)$;
3. $\forall\ A, B \subseteq X \quad rc(A \cup B) \geq rc(A) * rc(B)$.

and hence the mapping $\mathcal{B}_\tau : 2^X \to L$ defined by $\mathcal{B}_t au(A) = rc(A)$ is an L-valued bornology on the set X.

Theorem

Given two (Chang-Goguen) L-fuzzy topological spaces (X, τ_X), (Y, τ_Y) and a continuous mapping $f : (X, \tau_X) \to (Y, \tau_Y)$, the mapping $f : (X, \mathcal{B}\tau_X) \to (Y, \mathcal{B}\tau_Y)$ is bounded.

Thank you for your attention!

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