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Level dependent capacities-based Sugeno integrals

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References

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Capacity:
$(X, \mathcal{A})$, monotone m: $\mathcal{A} \rightarrow [0,1]$, $\mathcal{A}$-measurable functions $f : X \rightarrow [0,1]$,

$$S_m(f) = sup\ \{min\left(a, m(A)\right) \mid a \cdot 1_A \leq f\} \quad (1)$$

Equivalently, S_m can be expressed as

$$S_m(f) = sup\ \{min\left(a, m(f \geq a)\right) \mid a \in [0,1]\}, \quad (2)$$

or

$$S_m(f) = sup\ \{min\left(m(A),\ min(f(x) \mid x \in A)\right) \mid A \in \mathcal{A})\}. \quad (3)$$

In [2], another equivalent definition of Sugeno integral was introduced, namely

$$S_m(f) = inf\ \{max\left(a, m(f \geq a)\right) \mid a \in [0,1]\}. \quad (4)$$

$M\colon \mathcal{A} \times [0,1] \to [0,1]$; t ∈ $[0,1]$, $M(\cdot, t) = m_t$ is a capacity, is called a level dependent capacity.

$h_{m,f}\colon [0,1] \to [0,1]$, $h_{m,f}(t) = m(f \geq t)$, is decreasing,

$h_{M,f} : [0,1] \to [0,1]$, $h_{M,f}(t) = M(\{f \geq t\}, t) = m_t(f \geq t)$.

$$\left(h_{M,f}\right)_*(t) = inf\ \{h_{M,f}(u) |\ u \in\ ]0,t]\}$$

$$\left(h_{M,f}\right)^*(t) = sup\ \{h_{M,f}(v) |\ v \in [t,1]\}\ .$$

$$(Su_M)_*(f) = sup\ \left\{min\ \left(t, \left(h_{M,f}\right)_*(t)\right)\ |\ \ t \in [0,1]\ \right\} \qquad (5)$$

$$(Su_M)^*(f) = sup\ \left\{min\left(t,\ \left(h_{M,f}\right)^*(t)\ \right) |\ \ t \in [0,1]\ \right\} \qquad (6)$$

$$(Su_M)^*(f) = sup \left\{min \left(t, \left(h_{M,f}\right)^*(v)\right) \mid 0 \le t \le v \le 1\right\} \tag{7}$$

$$(Su_M)_*(f) = sup \left\{min \left(t, \left(h_{M,f}\right)_*(u)\right) \mid 0 \le u \le t \le 1 \right\} \tag{8}$$

$$Su_M^{(1)}(f) = sup \left\{min \left(a, m_a(A)\right) \mid a \cdot 1_A \le f \right\}, \tag{9}$$

$$Su_M^{(2)}(f) = sup \left\{min \left(a, m_a(f \ge a)\right) \mid a \in [0,1] \right\}, \tag{10}$$

$$Su_M^{(3)}(f) = sup\left\{min \left(t, m_t(A)\right) \mid A \in \mathcal{A}, t = inf(f(x) \mid x \in A)\right\} \tag{11}$$

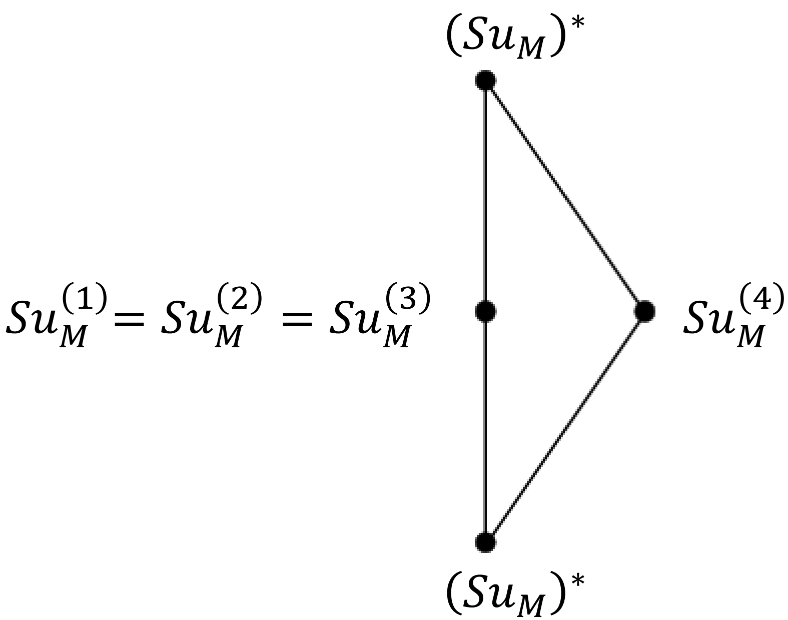
$$Su_M^{(4)}(f) = inf \left\{max \left(a, m_a(f \ge a)\right) \mid a \in [0,1]\right\}. \tag{12}$$

$Su_M^{(2)} = Su_M^{(3)}$ due to the monotonicity of $\mathrm{m_a}$.

$Su_M^{(1)} \geq Su_M^{(2)}$ because of $a \cdot 1_{\{f \geq a\}} \leq f$, and

$Su_M^{(1)} \leq Su_M^{(2)}$ because of

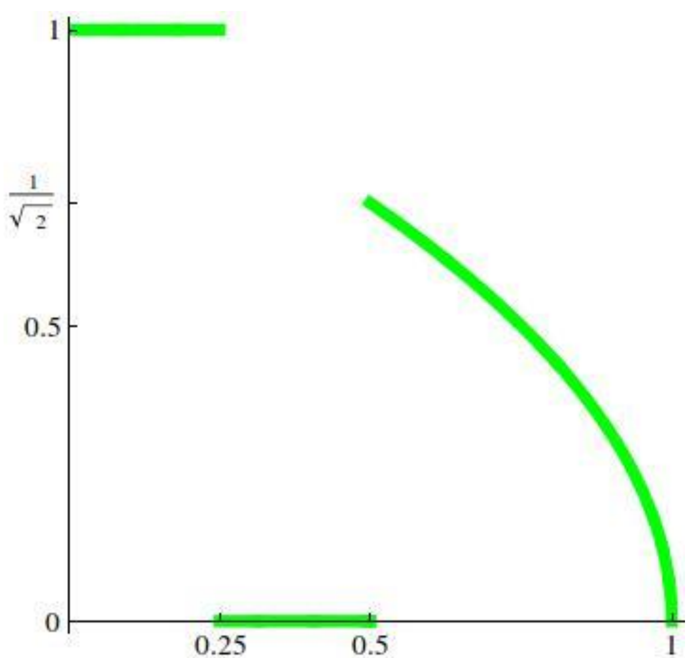
if $a \cdot 1_A \leq f$ then $A \subseteq \{f \geq a\}$. Thus $Su_M^{(1)} = Su_M^{(2)} = Su_M^{(3)}$.



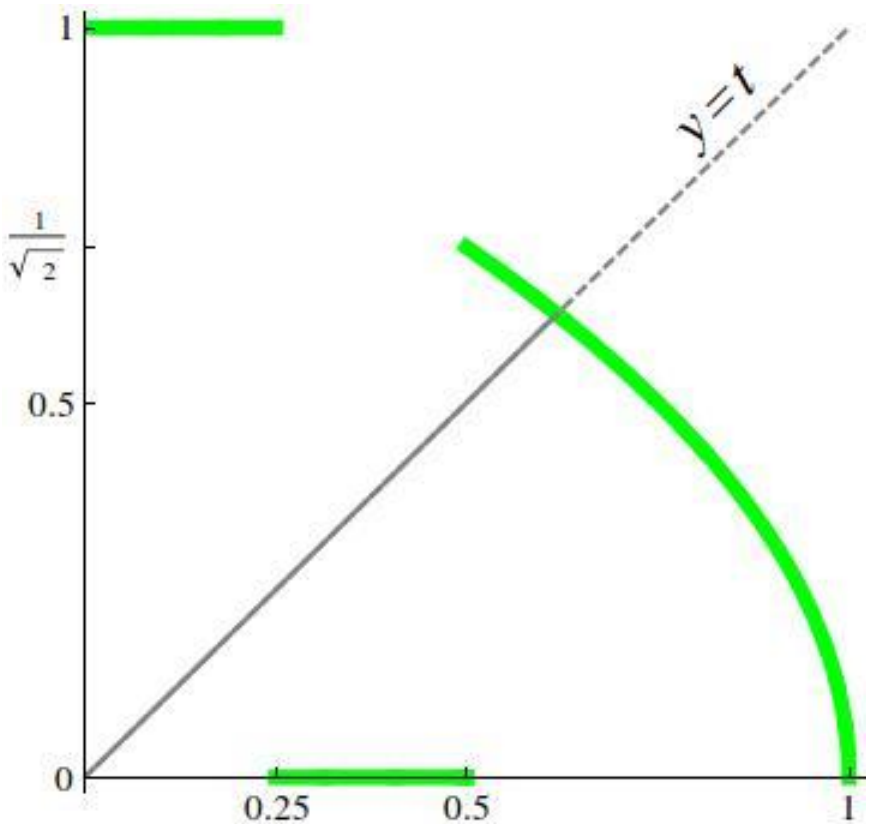
Example 1:

$(X, \mathcal{A}), X = [0,1]$ and $\mathcal{A} = \mathcal{B}([0,1])$ (Borel subsets of $[0,1]$).

$M = (m_t)_{t \in [0,1]}$: $m_t = m^*$ if $t \in \left[0, \frac{1}{4}\right], m_t = m_*$ if $t \in \left]\frac{1}{4}, \frac{1}{2}\right[$,
$m_t(A) = \sqrt{\lambda(A)}$; $A \in \mathcal{A}$, λ is the standard Lebesgue m. on $\mathcal{B}([0,1])$ if $t \in \left[\frac{1}{2}, 1\right]$,
$f_1(x) = x$.

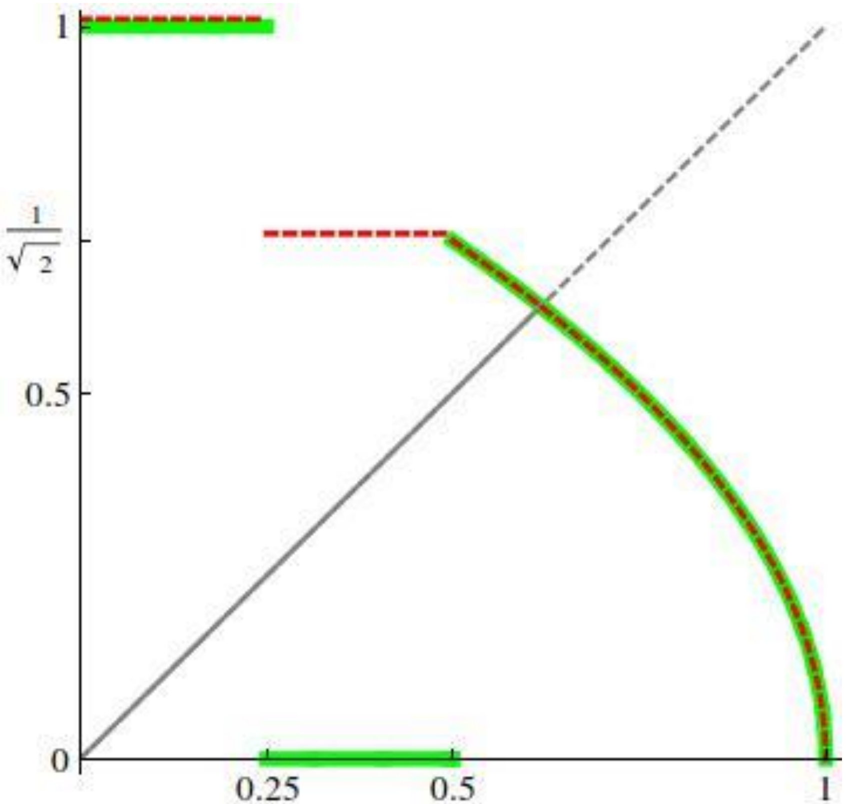


$$h_{M,f_1}(t) = \begin{cases} 1 \ if \ t \in \left[0, \frac{1}{4}\right], \\ 0 \ if \ t \in \left]\frac{1}{4}, \frac{1}{2}\right[, \\ \sqrt{1-t} \ if \ t \in \left[\frac{1}{2}, 1\right], \end{cases}$$



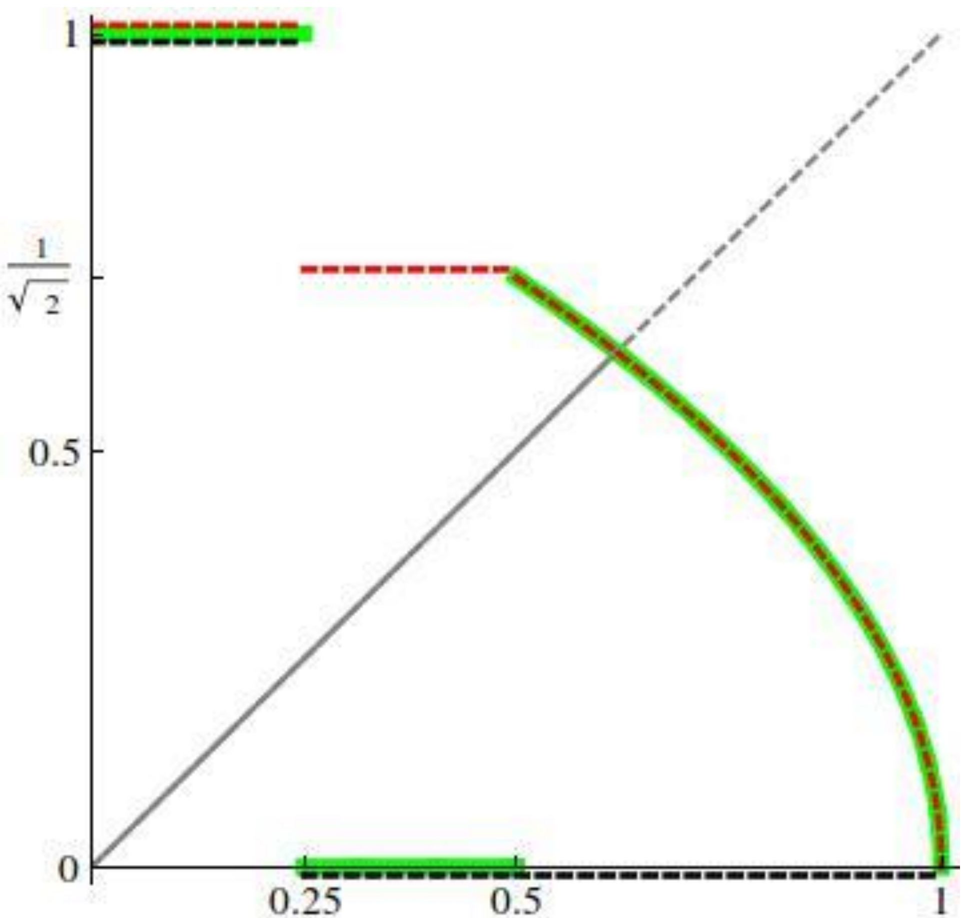
1
$\frac{1}{\sqrt{2}}$
0.5
0
0.25
0.5
1
y=t

$$\left(h_{M,f}\right)^*(t) = sup\ \{h_{M,f}(v) |\ v \in [t,1]\}$$



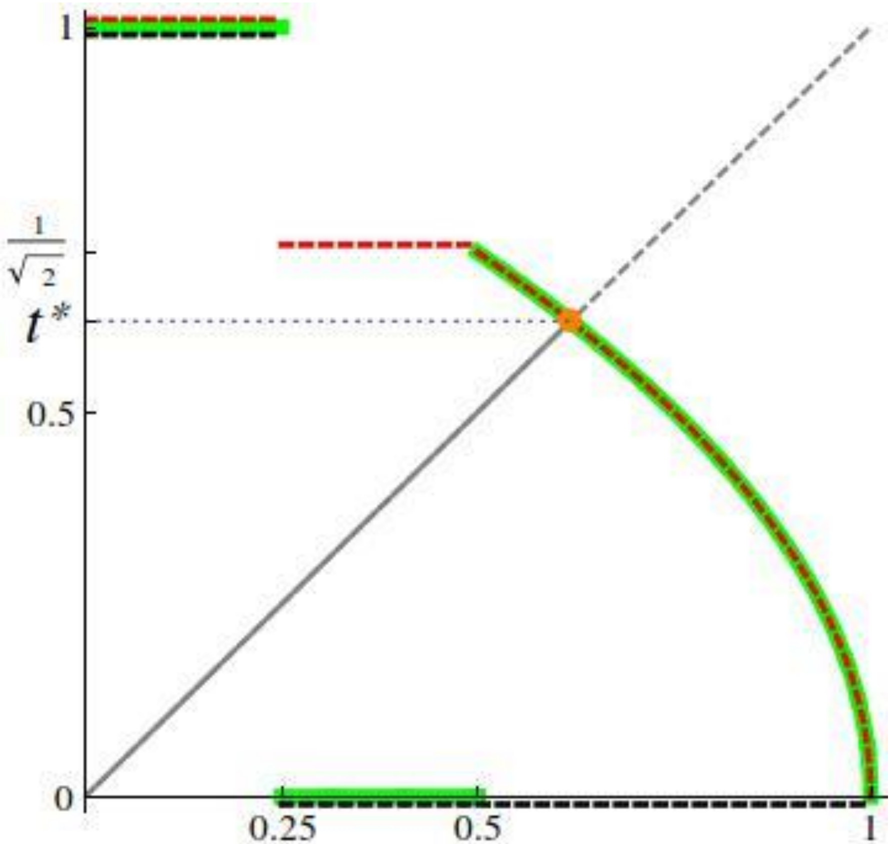
$$\left(h_{M,f_1}\right)^*(t) = \begin{cases} 1\ if\ t \in \left[0, \frac{1}{4}\right], \\ \sqrt{0.5}\ if\ t \in \left]\frac{1}{4}, \frac{1}{2}\right[, \\ \sqrt{1-t}\ if\ t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

$$\left(h_{M,f}\right)_*(t) = inf\ \{h_{M,f}(u) |\ u \in [0,t]\}$$



$$\left(h_{M,f_1}\right)_*(t) = \begin{cases} 1\ if\ t \in \left[0, \frac{1}{4}\right], \\ 0\ if\ t \in]\frac{1}{4}, 1]. \end{cases}$$

$$(Su_M)^*(f) = sup\ \{min\left(t, \left(h_{M,f}\right)^*(t)\right)\ \mid\ t \in [0,1]\}$$

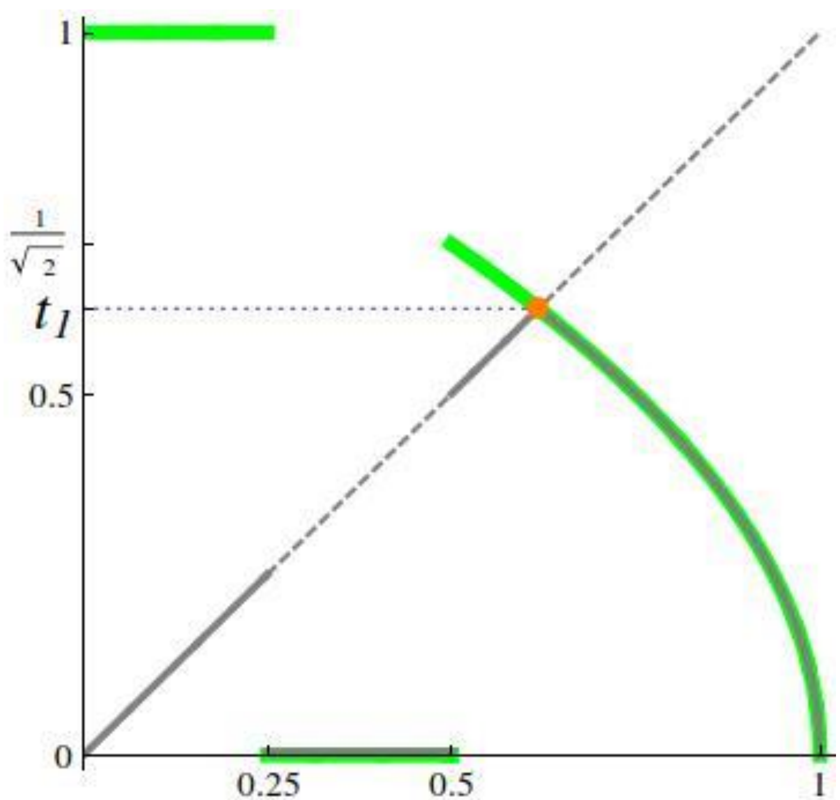


$$(\mathrm{Su_M})^*(f_1) = \frac{\sqrt{5}}{2} - \frac{1}{2} = \boldsymbol{t}^* = 0{,}618$$

$$Su_M^{(1)}(f) = sup\ \{min\left(a, m_a(A)\right) \mid a \cdot 1_A \leq f\ \}$$

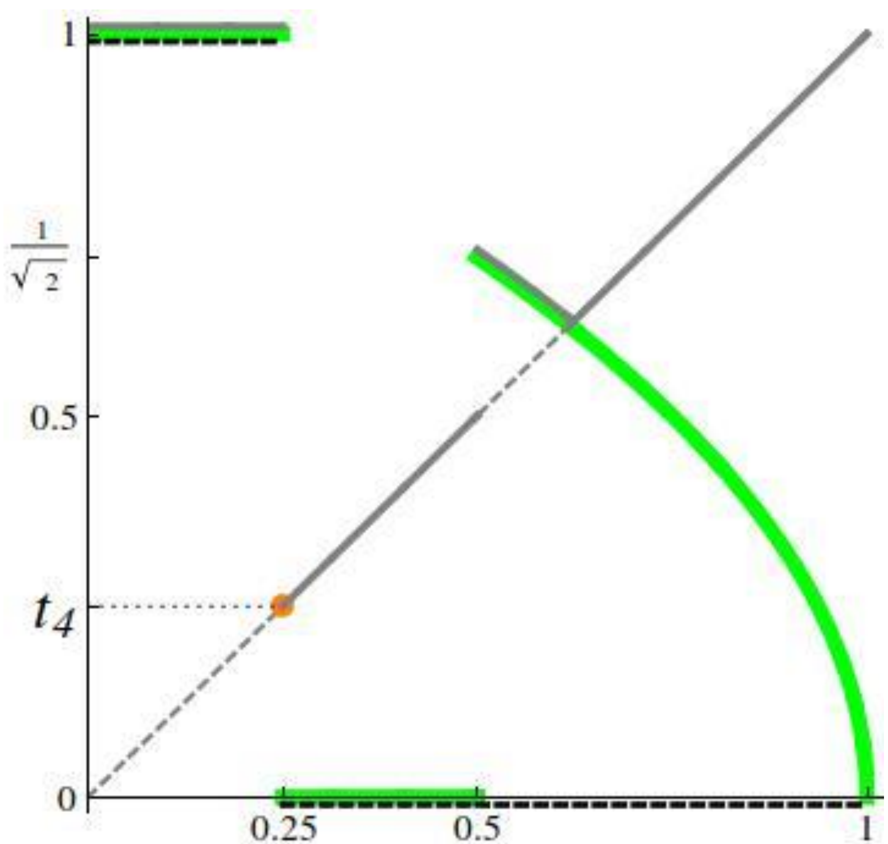
$$Su_M^{(2)}(f) = sup\ \{min\left(a, m_a(f \geq a)\right) \mid a \in [0,1]\ \}$$

$$Su_M^{(3)}(f) = sup\ \{min\left(t, m_t(A)\right) \mid A \in \mathcal{A}, t = inf(f(x) \mid x \in A)\}$$



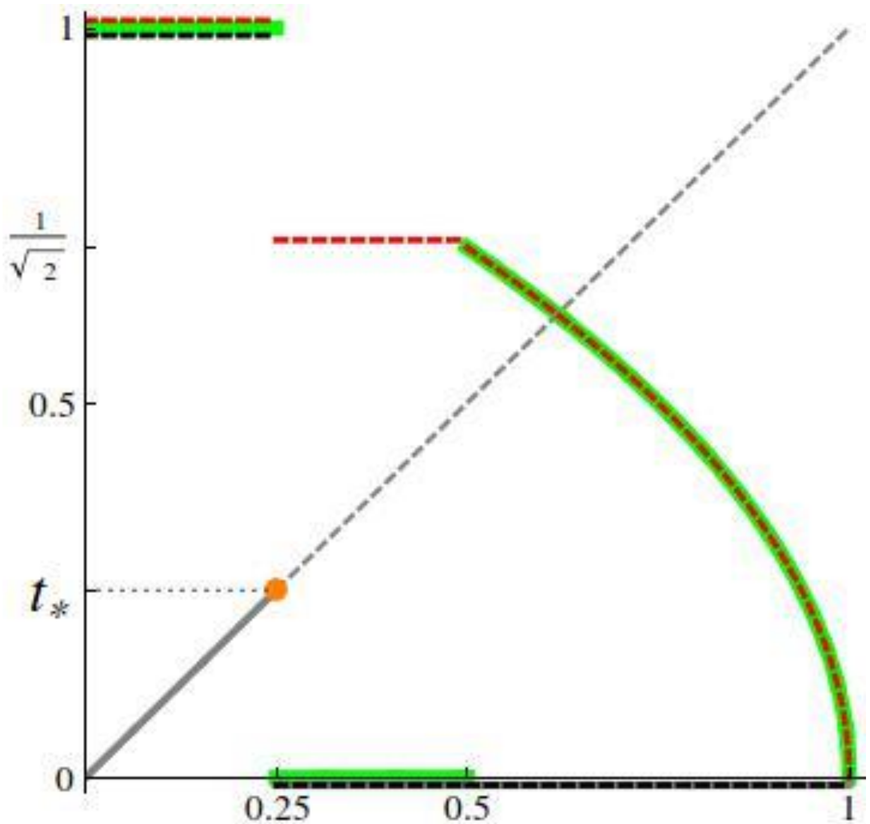
$$(\mathrm{Su_M})^*(f_1) = \frac{\sqrt{5}}{2} - \frac{1}{2} = Su_M^{(i)}(f_1), i = 1,2,3.$$

$$Su_M^{(4)}(f) = inf \ \{max\left(a, m_a(f \geq a)\right) \mid a \in [0,1]\}$$



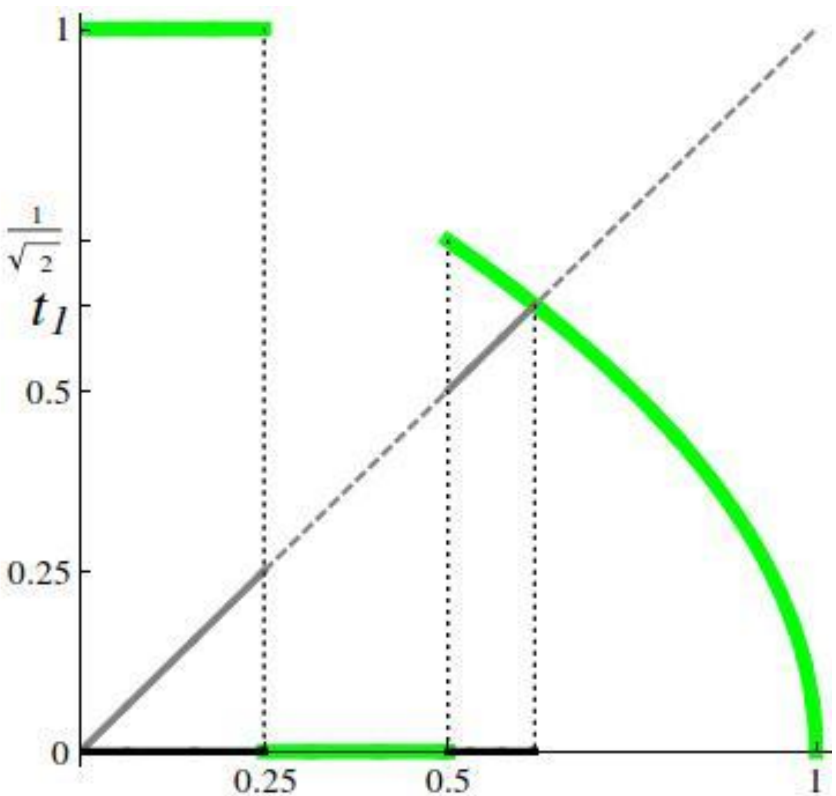
$$t_4 = \frac{1}{4} = Su_M^{(4)}(f_1)$$

$$(Su_M)_*(f) = sup\left\{min\left(t, \left(h_{M,f}\right)_*(t)\right) \mid t \in [0,1]\right\}$$

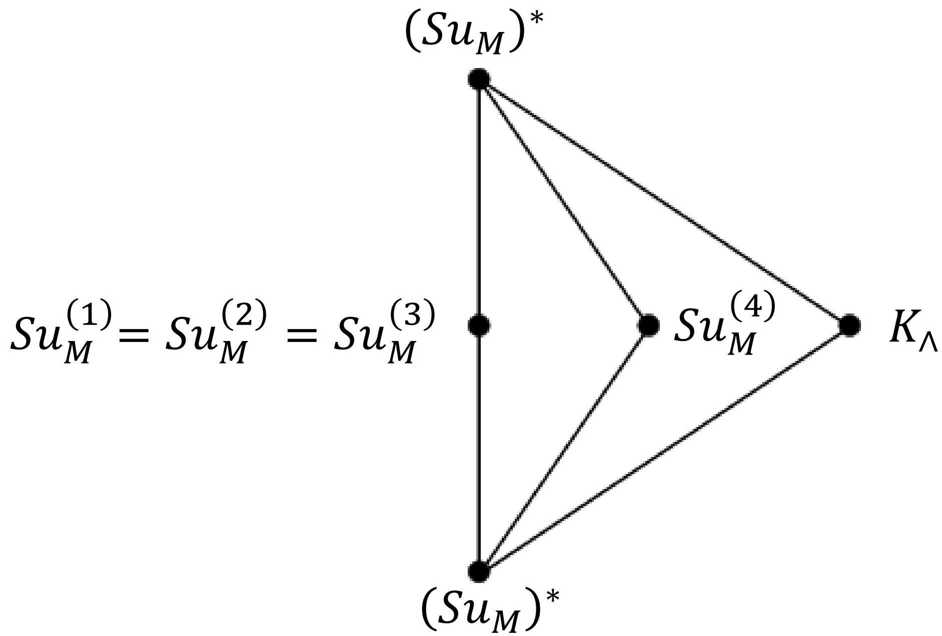


$$(\mathrm{Su_M})_*(f_1) = \frac{1}{4} = Su_M^{(4)}(f_1)$$

$$K_\wedge(m,f) = P_\wedge\big(\{(x,y) \in]0,1]^2 \mid y \leq h_{m,f}(x)\}\big)$$



$$K_\wedge(m,f_1) = \frac{\sqrt{5}}{2} - \frac{3}{4} = 0.368$$


$(Su_M)^*$
$Su_M^{(1)} = Su_M^{(2)} = Su_M^{(3)}$
$Su_M^{(4)}$
$K_\wedge$
$(Su_M)^*$

Thank you !

Example 2:

$X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, $\mathrm{f(x)} = \mathrm{x}$. For measurable space $(X, \mathcal{A})$ consider $M = (m_t)_{t \in [0,1]}$ given by

$$m_{\frac{1}{3}} = m_*, \qquad m_{\frac{2}{3}} = m^*, \qquad else\ m_t(A) = \lambda(A)$$

for $A \in \mathcal{A}$, where λ is the standard Lebesgue measure on $\mathcal{B}([0,1])$.
Then introduced integrals are :

$$(\mathrm{Su_M})_*(f) = \frac{1}{3} = Su_M^{(4)}(f), \quad (\mathrm{Su_M})^*(f) = \frac{2}{3} = Su_M^{(i)}(f), i = 1,2,3.$$

Example 2b):

Consider now $M = (m_t)_{t \in [0,1]}$ given by

$$m_0 = m_*, \; m_{\frac{1}{3}} = m^*, m_{\frac{2}{3}} = m_*, m_1 = m^*, else\ m_t(A) = t\ if\ A \notin \{\emptyset, X\}.$$

Then introduced integrals are :

$$(\mathrm{Su_M})_*(f) = 0 = Su_M^{(4)}(f), \quad (\mathrm{Su_M})^*(f) = 1 = Su_M^{(i)}(f), i = 1,2,3.$$