Triangle Functions

Constructions and Functional (In)equalities

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Motivation and interpretation

Distance distribution functions

Distance distribution functions

A distribution function is a function $F\colon \overline{\mathbb{R}} \to [0,1]$ such that

- F is increasing;
- F is left-continuous on $\mathbb{R}$;
- $F(-\infty) = 0$ and $F(\infty) = 1$.

The set of all distributions functions (d.f.) will be denoted by Δ.

A distance distribution function is a distribution function F such that $F(0) = 0$.

The set of all distance distribution functions (d.d.f.) will be denoted by Δ^+.

Possible interpretations of elements of Δ^+

- In **probabilistic metric spaces** a function $\mathcal{F}$ assigns to each pair of elements p and q of some non-empty set X a distance distribution function.

  Then, for all $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between p and q is less than x.

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  Then, for all $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between p and q is less than x.
- A **finite fuzzy real** is a fuzzy subset A of $\mathbb{R}$ such that
  - A is increasing;
  - A is left-continuous (on $\mathbb{R}$);
  - $\inf\{A(x) \mid x \in \mathbb{R}\} = 0$ and $\sup\{A(x) \mid x \in \mathbb{R}\} = 1$.

  A **non-negative finite fuzzy real** is a finite fuzzy real A such that $A(0) = 0$.

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  A function $F \in \Delta^+$ may be interpreted as a fuzzy set modeling, e.g., "at least 5"
- Interpretation of distribution functions of type $1 - F$.
- Interpretation of Δ^+ as a **bounded lattice**.

The lattice structure of Δ^+

Definition

- For all $F, G \in \Delta^+$:

$$F \leq G \quad :\Leftrightarrow \quad \forall x \in \overline{\mathbb{R}} : F(x) \leq G(x).$$

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- Top and bottom element:

$$\varepsilon_0(x) := \begin{cases} 0, & x = 0, \\ 1, & x > 0, \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) := \begin{cases} 0, & x < \infty, \\ 1, & x = \infty, \end{cases}$$

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$\dashrightarrow$ **$(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$ is a bounded lattice.**

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  $\dashrightarrow$ Closedness w.r.t. operations induced by the **extension principle**, i.e.

$$A \circledast B(x) = \sup\{T(A(u), B(v)) \mid u, v \in \mathbb{R}, u * v = z\}$$

with T some t-norm and $*$ a binary operation on $\mathbb{R}$.

Triangle functions

(Šerstnev, 1964)

Definition

A binary operation $\tau\colon \Delta^+ \times \Delta^+ \to \Delta^+$ which fulfills for all $F, G, H \in \Delta^+$

(i) $\tau(F, \varepsilon_0) = F$,

(ii) $\tau(F, G) \succeq \tau(F, H)$ whenever $G \succeq H$,

(iii) $\tau(F, G) = \tau(G, F)$,

(iv) $\tau(F, \tau(G, H)) = \tau(\tau(F, G), H)$.

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$$(\Delta^+, \tau, \leq)$$

is a commutative, partially ordered semigroup with neutral element ε_0

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is called a triangle function.

τ is a **triangular norm**
on the bounded lattice $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$.

Examples of triangle functions

- Consider some left-continuous t-norm T.

$$\tau_T(F,G)(x) = \sup\{T(F(u),G(v)) \mid u+v=x\}$$

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- Consider some ordinal sum C of product summands only.

$$\sigma_C(F,G)(x) = \int_{\{(u,v)|u+v<x\}} dC(F(u),G(v))$$

How to construct triangle functions?

Approach 1

Are there approaches similar to the construction/representation of (continuous) t-norms on $[0, 1]$?

T-norms on $[0, 1]$

Triangle functions and t-norms

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Constructions of t-norms on $[0,1]$

- Ordinal sums and extensions
- Additive generators

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Constructions of t-norms on $[0,1]$

- Ordinal sums and extensions
- Additive generators

$\dashrightarrow$ **similar approaches for triangle functions?**

(Still) open problems by Schweizer and Sklar

Problem 7.9.1 (Schweizer, Sklar, 1983)

"... In particular determine all continuous triangle functions and, if possible, find a representation corresponding to the one given in Theorems 5.3.8 and 5.4.1."

Theorem 5.3.8: Representation as minimum, Archimedean t-norm, or ordinal sum thereof;
Theorem 5.4.1: Representation by generators;

Problem 7.9.5 (Schweizer, Sklar, 1983)

" Suppose that T is a continuous t-norm. ... In particular, if T is an ordinal sum is $\tau_{T,L}$ an ordinal sum?"

Approach 2

What can we infer from the construction methods for t-norms on lattices?

Extensions of t-norms on bounded lattices (Sam,Kle,Mes 2008)

Consider

- a bounded lattice $(L, \leq, 0_L, 1_L)$,
- a bounded (and complete) sublattice $(S, \leq, a, b)$, and
- a t-norm $T^S\colon S^2 \to S$ on S.

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Weakest and strongest extension

Determine operations $\underline{T}^L, \overline{T}^L \colon L^2 \to L$ such that

- $\underline{T}^L|_{S^2} = \overline{T}^L|_{S^2} = T^S$;
- for all t-norms $T \colon L^2 \to L$ with $T|_{S^2} = T^S$:

$$\underline{T}^L \leq T \leq \overline{T}^L;$$

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$$\underline{T}^L \leq T \leq \overline{T}^L;$$

- $\underline{T}^L$ resp. $\overline{T}^L$ is a t-norm on L.

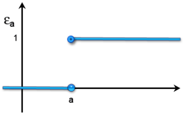
Interesting sublattices of Δ^+

- Step functions $\varepsilon_a := \mathbf{1}_{]a,\infty]}$ for all $a \geq 0$;

  $$E^+ = \{\varepsilon_a \mid a \geq 0\};$$

  $$\varepsilon_a \geq \varepsilon_b \Leftrightarrow a \leq b.$$

  $$(E^+, \leq, \varepsilon_\infty, \varepsilon_0)$$



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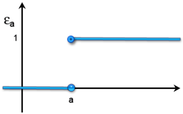
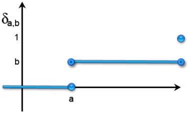


- "Constant functions"
  $\delta_{a,b} := b\varepsilon_a + (1-b)\varepsilon_\infty$ for all $a \geq 0, b \in [0,1]$;

$$\Delta^+_\delta = \{\delta_{a,b} \mid a \geq 0, b \in [0,1]\};$$

$$\delta_{s,t} \leq \delta_{s,u} \Leftrightarrow t \leq u \Leftrightarrow \delta_{t,s} \geq \delta_{u,s}.$$

$$(\Delta^+_\delta, \leq, \delta_{a,0}, \delta_{1,0})$$



Strongest extension

The model

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Define $T\colon L^2 \to L$, for all $x, y \in L$ by

$$T(x,y) = \begin{cases} T^S(x,y), & \text{if } (x,y) \in S^2, \\ x \wedge y, & \text{otherwise} \end{cases}$$

If T is a t-norm, then $T = \overline{T}^L_{T^S}$.

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The particular case $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$

For any non-trivial bounded sublattice $(S, \leq, \varepsilon_\infty, \varepsilon_0)$, T is not a t-norm for some t-norm T^S on S.

Weakest extension

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Define

- $T^{S\cup\{0,1\}} \colon (S \cup \{0,1\})^2 \to (S \cup \{0,1\})$ by

$$T^{S\cup\{0,1\}}(x,y) := \begin{cases} x \wedge y, & \text{if } 1 \in \{x,y\}, \\ 0, & \text{if } 0 \in \{x,y\}, \\ T(x,y), & \text{if } (x,y) \in S^2. \end{cases}$$

- $\underline{T}^L_{T^S} \colon L^2 \to L$ by

$$\underline{T}^L_{T^S} := \begin{cases} x \wedge y, & \text{if } 1 \in \{x,y\}, \\ T^{S\cup\{0,1\}}(x^*, y^*), & \text{otherwise}, \end{cases}$$

with $x^* = \sup\{z \mid z \leq x, z \in S \cup \{0,1\}\}$,

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  with $x^* = \sup\{z \mid z \leq x, z \in S \cup \{0,1\}\}$,

then $\underline{T}^L_{T^S}$ is a t-norm and the smallest possible extension of T^S.

Weakest extension

The particular case $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$

For $L = (\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$, each bounded and complete sublattice is appropriate for the weakest extension.

Approach 3
Look for other strategies!

Historic examples of triangle functions

- Consider some left-continuous t-norm T.

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) \mid u + v = x\}$$

- Convolution

$$(F * G)(x) = \int_{[0,x[} F(x - t) dG(t)$$

- Consider some ordinal sum C of product summands only.

$$\sigma_C(F, G)(x) = \int_{\{(u,v) \mid u+v<x\}} dC(F(u), G(v))$$

- Consider some left-continuous t-norm T.

$$\pi_T(F, G)(x) = T(F(x), G(x))$$

Types of triangle functions

General remarks

Consider a binary operation on Δ^+, i.e. ,

$$\Theta\colon \Delta^+ \times \Delta^+ \to \Delta^+, \quad (F, G) \mapsto \Theta(F, G).$$

What shall/can we do to determine the value $\Theta(F, G)(x)$?

- Strategy 1: Pointwise induced triangle functions;
- Strategy 2: “Splitting the argument”
  - involving semicopulas;
  - involving co-semicopulas;
  - involving quasi-copulas;
- Strategy 3: “Involving measures and integrals”.

Pointwise induced operations

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Strategy 1: “Passing through the argument”

- Consider some binary function A on $[0, 1]$;
- Determine $\Theta(F, G)(x)$ by

$$\Theta(F, G)(x) = A(F(x), G(x)).$$

$\dashrightarrow$ Pointwise induced operations π_A

Pointwise induced operations

Theorem

Consider a function $A\colon [0,1]^2 \to [0,1]$.

$$\pi_A \text{ is a binary operation on } \Delta^+.$$
$$\Leftrightarrow$$
$$A \text{ is a left–continuous binary aggregation operator.}$$

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Theorem

Consider a function $T\colon [0,1]^2 \to [0,1]$.

$$\pi_T \text{ is a triangle function.}$$
$$\Leftrightarrow$$
$$T \text{ is a left–continuous t-norm.}$$

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Strategy 2: “Splitting the argument”

- Consider
  - some binary function A on $[0, 1]$,
  - some binary operation L on $\overline{\mathbb{R}}^+$,
  - choose $\Omega = \sup$ or $\Omega = \inf$.
- Determine $\Theta(F, G)(x)$ by

$$\Theta_{A,L,\Omega}(x) = \Omega\{A(F(u), G(v)) \mid L(u, v) = x\}$$

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$\dashrightarrow$ Operations and triangle functions of the type $\tau_{f,L}(\tau_{T,L})$, $\tau^*_{S^*,L}(\tau_{T^*,L})$, $\rho_{Q,L}(\rho_{T,L})$

Operations on $\overline{\mathbb{R}}^+$

The class $\mathfrak{L}$

We consider binary operations L on $\overline{\mathbb{R}}^+$ such that

- L is surjective, i.e., $\mathrm{Ran}_L = \overline{\mathbb{R}}^+$,
- L is increasing in each place,
- L is continuous on $\overline{\mathbb{R}}^+$ except possibly at the points $(0, \infty)$ and $(\infty, 0)$.

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Additional properties

Consider some $L \in \mathfrak{L}$:

(LS) L fulfills for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}^+$ with $u_1 < u_2, v_1 < v_2$

$$L(u_1, v_1) < L(u_2, v_2).$$

(L0) L has 0 as its neutral element.

The class $\tau_{f,L}$

Definition

Consider some $L \in \mathfrak{L}$ and a function $f\colon [0,1]^2 \to [0,1]$.

Define $\tau_{f,L} : \Delta^+ \times \Delta^+ \to [0,1]^{\overline{\mathbb{R}}^+}$ by

$$\tau_{f,L}(F,G)(x) = \sup\{f(F(u),G(v)) \mid L(u,v) = x\}.$$

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Theorem

Assume additionally that L satisfies (LS) and (L0).

- If $\tau_{T,L}$ is a triangle function, then T is a t-norm.

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Assume additionally that L satisfies (LS) and (L0).

- If $\tau_{T,L}$ is a triangle function, then T is a t-norm.
- If L is commutative and associative and if T is a left-continuous t-norm, then $\tau_{T,L}$ is a triangle function.

The class $\tau^*_{S^*,L}$

Definition

Consider some $L \in \mathfrak{L}$ and a co-semicopula $S^*\colon [0,1]^2 \to [0,1]$, i.e.,

$$S(x,y) = 1 - S^*(1-x, 1-y)$$

is a semicopula.

Define $\tau^*_{S^*,L}\colon \Delta^+ \times \Delta^+ \to [0,1]^{\overline{\mathbb{R}}^+}$ by

$$\tau^*_{S^*,L}(F,G)(x) = \inf\{S^*(F(u), G(v)) \mid L(u,v) = x\}.$$

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Theorem

Assume additionally that L is commutative, associative and satisfies (LS) and (L0).

$$T^* \text{ is a continuous t-conorm.}$$
$$\Rightarrow$$
$$\tau^*_{T^*,L} \text{ is a triangle function.}$$

The class $\sigma_{C,L}$

General remarks

Consider a binary operation on Δ^+, i.e. ,

$$\Theta\colon \Delta^+ \times \Delta^+ \to \Delta^+, \quad (F, G) \mapsto \Theta(F, G).$$

What shall/can we do to determine the value $\Theta(F, G)(x)$?

Strategy 3: “Involving measures and integrals”, e.g.

- Consider
  - some copula C,
  - some binary operation L on $\overline{\mathbb{R}}^+$.
- Determine $\Theta(F, G)(x)$ by

$$\Theta(F, G)(x) = \int_{L(x)} dC(F(u), G(v))$$

with $L(x) = \{(u, v) \mid L(u, v) < x\}$.

Involving copulas

Definition

Consider some $L \in \mathfrak{L}$ and some copula C.

Define the function $\sigma_{C,L} \colon \Delta^+ \times \Delta^+ \to \Delta^+$ by

$$\sigma_{C,L}(F,G)(0) := 0, \qquad \sigma_{C,L}(F,G)(\infty) := 1$$

and

$$\sigma_{C,L}(F,G)(x) := \int\limits_{L(x)} dC(F(u),G(v))$$

for all $x \in]0,+\infty[$, where

$$L(x) = \{(u,v) \mid u,v \in \mathbb{R}^+, L(u,v) < x\}.$$

Properties

Neutral element

Consider some $L \in \mathfrak{L}$ and some copula C.

$$\sigma_{C,L} \text{ has } \varepsilon_0 \text{ as its neutral element.}$$
$$\Leftrightarrow$$
$$L \text{ fulfills (L0).}$$

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(Frank, 1975, Frank 1991)

Consider some $L \in \mathfrak{L}$ and some copula C.

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$$\sigma_{C,L} \text{ is commutative.} \Leftarrow C \text{ and } L \text{ are commutative.}$$

Properties

Associativity

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Particular $L \in \mathfrak{L}$

Consider some $L \in \mathfrak{L}$ satisfying (L0) and (LS) and being commutative and associative.

- $L = \max$;
- there exists some continuous and strictly increasing function $h\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ with

$$L(u, v) = h^{-1}(h(u) + h(v)).$$

Characterization

Theorem (Frank, 1991)

Consider

- some $L \in \mathfrak{L}$ such that $L(u, v) = h^{-1}(h(u) + h(v))$ for some a continuous, strictly icnreasing function $h \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$;
- some copula C.

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- some copula C.

$$\sigma_{C,L} \text{ is associative.}$$
$$\Leftrightarrow$$
C is a (trivial or non-trivial) ordinal sum of product t-norms.

Summary

We have discussed different strategies for constructing triangle functions.

Are there triangle functions different from the types we have listed and presented above?

Functional (in)equalities

Cauchy's functional equation

Cauchy's functional equation

Cauchy's functional equation for triangle functions

Consider a triangle function τ.

A mapping $\varphi\colon \Delta^+ \to \Delta^+$ is a solution of the Cauchy's functional equation for τ, if, and only if, for all $F, G \in \Delta^+$,

$$\varphi\left(\tau(F, G)\right) = \tau\left(\varphi(F), \varphi(G)\right).$$

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Results for triangle functions achieved by T. Riedel based on work by R.C. Powers.

Cauchy's functional equation
Necessary conditions

Let φ be a solution of the Cauchy equation for a triangle function τ. Then the following holds:

- **Idempotent elements**: For all $F \in \Delta^+$ with $\tau(F, F) = F$, it holds that

$$\varphi(F) = \varphi(\tau(F, F)) = \tau(\varphi(F), \varphi(F)),$$

i.e., φ maps idempotents to idempotents.
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  In particular:

- **Neutral element**:

$$\varphi(\varepsilon_0) = \varphi(\tau(\varepsilon_0, \varepsilon_0)) = \tau(\varphi(\varepsilon_0), \varphi(\varepsilon_0)),$$

  i.e., $\varphi(\varepsilon_0)$ is an idempotent element of τ.

Cauchy's functional equation

Some solutions

Consider an arbitrary triangle function τ and denote by Id_τ its set of idempotent elements.

Then the following functions $\varphi\colon \Delta^+ \to \Delta^+$ are solutions of the Cauchy's functional equation w.r.t. τ:

- **Constant functions**: For all $H \in \mathrm{Id}_\tau$, the (constant) functions φ defined, for all $F \in \Delta^+$, by

$$\varphi(F) = H.$$

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- **Functions φ_H** defined, for arbitrary $H \in \mathrm{Id}_\tau$ and all $F \in \Delta^+$, by

$$\varphi_H(F) = \tau(F, H).$$

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In particular with $H = \varepsilon_0$, $\varphi_{\varepsilon_0} = \mathrm{id}_{\Delta^+}$.
- **Powers τ^n of τ**, defined, for all $n \in \mathbb{N}$, $n \geq 2$, and all $F \in \Delta^+$, by
$$\varphi(F) = \tau^n(\underbrace{F, \ldots, F}_{n \text{ times}}) = \tau(\tau^{n-1}(\underbrace{F, \ldots, F}_{n-1 \text{ times}}), F).$$

Cauchy's functional equation

Characterization for sup-continuous functions (Riedel, 1991)

Theorem

Consider a sup-continuous triangle function τ and a sup-continuous function $\varphi\colon \Delta^+ \to \Delta^+$.

Then φ is a solution of the Cauchy's equation if and only if

$$\varphi\left(\tau\left(\delta_{a,b}, \delta_{c,d}\right)\right) = \tau\left(\varphi\left(\delta_{a,b}\right), \varphi\left(\delta_{c,d}\right)\right),$$

for all a and c in $\overline{\mathbb{R}}_+$, and for all b and d in $[0,1]$, where

$$\delta_{a,b}(x) := \begin{cases} 0, & x \in [0,a], \\ b, & x \in \,]a,+\infty[\,, \\ 1, & x = +\infty. \end{cases}$$

Cauchy's functional equation
Remarks

- Due to Tardiff, 1975:
  - If T is continuous, then τ_T is sup-continuous.
  - However, not all triangle functions are sup-continuous, e.g., convolution, $\sigma_\prod$ is not.

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The strict case

φ is a solution if and only if there exists $k, l > 0$ such that, for all $x \in \mathbb{R}$ and all $F \in \Delta^+$

$$\varphi(F)(x) = g^{-1}\left(k \cdot g(F(l \cdot x))\right)$$

with g the additive generator of the strict t-norm T.

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with g the additive generator of the strict t-norm T.

- Generalization of the results for $\tau_{T,L}$ for some generated L, i.e., $L(x, y) = f^{-1}(f(x) + f(y))$ in Riedel, 1992.

Cauchy's functional equation
Summary

Cauchy's functional equation for triangle functions has been studied only for triangle functions of the type

$$\tau = \pi_T, \quad \tau = \tau_T, \quad \tau = \tau_{T,L},$$

with restriction on both the t-norm T and on the function L.

What are the solution when τ belongs to a different family of triangle functions?

Functional inequality: Dominance

Definition by Schweizer and Sklar

Definition (Schweizer, Sklar, 1983)

Consider a partially ordered set $(P, \leq)$ and two associative binary operations f, g on P with common identity e.

Then f **dominates** g ($f \gg g$) if, for all x, y, u, v in P,

$$f(g(x, y), g(u, v)) \geq g(f(x, u), f(y, v)).$$

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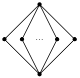
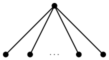
Remarks:

- Due to the common neutral elemtent, dominance of f over g induces that $f \geq g$.
- Associativity and commutativity of an operation f ensures its self-dominance (bisymmetry).
- Dominance does not constitute a transitive relation on the set of all triangle functions.

The (hi)story of dominance of t-norms

Known results on families of t-norms
until 2005

Family of t-norms	$T_\lambda \gg T_\mu$	Hasse-Diag
Schweizer-Sklar $(T^{\mathbf{SS}}_\lambda)_{\lambda\in[-\infty,\infty]}$ (Sherwood, 1984)	$\lambda \leq \mu$	
Aczél-Alsina $(T^{\mathbf{AA}}_\lambda)_{\lambda\in[0,\infty]}$ Dombi $(T^{\mathbf{D}}_\lambda)_{\lambda\in[0,\infty]}$ Yager $(T^{\mathbf{Y}}_\lambda)_{\lambda\in[0,\infty]}$ (Klement, Mesiar, Pap, 2000)	$\lambda \geq \mu$	
Frank $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$ Hamacher $(T^{\mathbf{H}}_\lambda)_{\lambda\in[0,\infty]}$ (Sarkoci, 2005)	$\lambda = 0$, $\lambda = \mu$, $\mu = \infty$	
Mayor-Torrens $(T^{\mathbf{MT}}_\lambda)_{\lambda\in[0,1]}$ Dubois-Prade $(T^{\mathbf{DP}}_\lambda)_{\lambda\in[0,1]}$ (Sam, De Baets, De Meyer, 2005)	$\lambda = 0, \lambda = \mu$	

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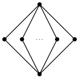
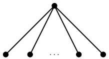
Frank $(T_\lambda^{\mathbf{F}})_{\lambda\in[0,\infty]}$ Hamacher $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,\infty]}$ (Sarkoci, 2005)	$\lambda = 0,\ \lambda = \mu,\ \mu = \infty$	
Mayor-Torrens $(T_\lambda^{\mathbf{MT}})_{\lambda\in[0,1]}$ Dubois-Prade $(T_\lambda^{\mathbf{DP}})_{\lambda\in[0,1]}$ (Sam, De Baets, De Meyer, 2005)	$\lambda = 0, \lambda = \mu$	

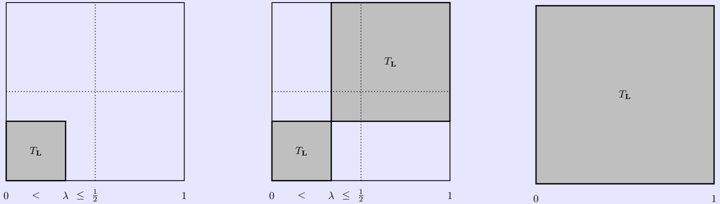
On all these families of t-norms, dominance is **transitive** and therefore an **order relation**.

Non-transitivity of dominance for continuous t-norms

Counter-example

(Sarkoci, 2008)

Let $\lambda \in \left[0, \frac{1}{2}\right]$.



$T_\lambda^{\mathbf{MT}} = (\langle 0, \lambda, T_{\mathbf{L}} \rangle)$ $\quad$ $T_\lambda = (\langle 0, \lambda, T_{\mathbf{L}} \rangle, \langle \lambda, 1, T_{\mathbf{L}} \rangle)$ $\quad$ $T_{\mathbf{L}}$

Then

$$T_\lambda^{\mathbf{MT}} \gg T_\lambda, \quad T_\lambda \gg T_{\mathbf{L}}, \quad T_\lambda^{\mathbf{MT}} \not\gg T_{\mathbf{L}}.$$

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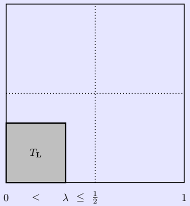

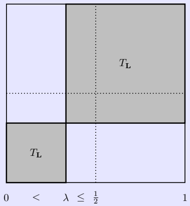

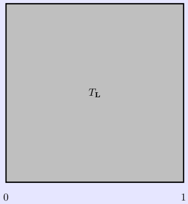

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Dominance is not transitive
on the class of (continuous) t-norms.

Generalized Mulholland inequality (SamPla, De Baets, De Meyer, 2008)

Sufficient and necessary conditions

Consider two continuous Archimedean t-norms T_1 and T_2 with additive generators t_1 and t_2. If the function

$$h = t_1 \circ t_2^{(-1)} \colon \ [0, \infty] \to [0, \infty]$$

is convex on $]0, t_2(0)[$ and . . .

$$\begin{array}{ccc}
\begin{array}{c} h' \text{ exists and} \\ h' \text{ is log-convex on }]0, t_2(0)[\end{array} & \Rightarrow & \begin{array}{c} h' \text{ exists and} \\ h' \text{ is geo-convex on }]0, t_2(0)[\end{array} \\
 & & \Downarrow \\
h \text{ is log-convex on }]0, t_2(0)[& \Rightarrow & h \text{ is geo-convex on }]0, t_2(0)[\\
 & & \Downarrow \\
 & & h \text{ fulfills } \boxed{\text{gen. Mulholland}} \\
 & & \Updownarrow \\
 & & T_1 \gg T_2
\end{array}$$

Easy-to-check-conditions ($T_1 \gg T_2$) (SamPla, De Baets, De Meyer, 2009)

Conditions on h

Consider two continuous Archimedean t-norms T_1 and T_2 with sufficiently often differentable additive generators t_1 and t_2. Define the function $h = t_1 \circ t_2^{(-1)}\colon\ [0,\infty] \to [0,\infty]$. Then

- h is **convex** on $]0, t_2(0)[$, if and only if, for all $u \in\,]0,1[$,

$$t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \geq 0.$$

- h is **log-convex** on $]0, t_2(0)[$, if and only if, for all $u \in\,]0,1[$,

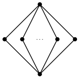
$$t_1'^2(u)t_2'(u) + t_1(u)\left(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)\right) \geq 0.$$

- h is **geo-convex** on $]0, t_2(0)[$, if and only if, for all $u \in\,]0,1[$,

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \geq \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)}$$

New results on families of t-norms (SamPla, 2009)

All families are taken from the book on associative functions by Alsina, Frank, Schweizer, 2006 resp. the book on copulas by Nelsen, 2006.

Family of t-norms		$T_\lambda \gg T_\mu$	Hasse-Diag
$(T^{8}_{\lambda})_{\lambda\in[0,\infty]}$	log-convexity of h'	$\lambda \leq \mu$	
$(T^{15}_{\lambda})_{\lambda\in[0,\infty]}$	geo-convexity of h	$\lambda \leq \mu$	
$(T^{22}_{\lambda})_{\lambda\in[0,\infty]}$	geo-convexity of h'	$\lambda \leq \mu$	
$(T^{23}_{\lambda})_{\lambda\in[0,\infty]}$	geo-convexity of h	$\lambda \leq \mu$	
$(T^{9}_{\lambda})_{\lambda\in[0,\infty]}$		$\lambda=\infty,\ \lambda=\mu,\ \mu=0$	

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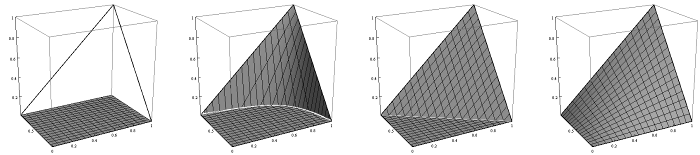
The family of Sugeno-Weber t-norms

The family of Sugeno-Weber t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda \in [0,\infty]}$ is given by

$$T_\lambda^{\mathbf{SW}}(u,v) = \begin{cases} T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \max(0, (1-\lambda)uv + \lambda(u+v-1)), & \text{if } \lambda \in \,]0,\infty[\,. \end{cases}$$



$T_\infty^{\mathbf{SW}}$ $T_{10}^{\mathbf{SW}}$ $T_1^{\mathbf{SW}}$ $T_0^{\mathbf{SW}}$

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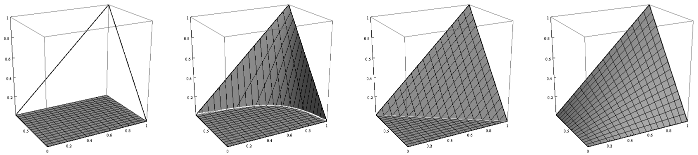


$T_\infty^{\mathbf{SW}}$ $T_{10}^{\mathbf{SW}}$ $T_1^{\mathbf{SW}}$ $T_0^{\mathbf{SW}}$

Results on dominance

- Partial results based on the differential sufficient conditions
- Full characterization now available (proven by CAD)

The family of Sugeno-Weber t-norms (Kauers, Pillwein, SamPla, 2011)

Theorem

Consider the family of Sugeno-Weber t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$.

$T_\lambda^{\mathbf{SW}}$ dominates $T_\mu^{\mathbf{SW}}$ if and only if one of the following holds:

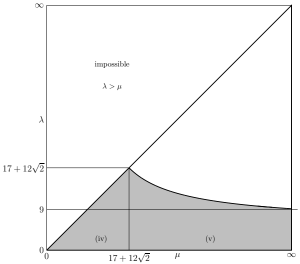
(i) $\lambda = 0$,

(ii) $\mu = \infty$,

(iii) $\lambda = \mu$,

(iv) $0 < \lambda < \mu \leq 17 + 12\sqrt{2}$,

(v) $17 + 12\sqrt{2} < \mu$ and $0 < \lambda \leq \left(\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}}\right)^2$.



Dominance for triangle functions

Why dominance for functions on Δ^+?

In probabilistic metric (PM) spaces a function $\mathcal{F}$ assigns to each pair of elements p and q in a non-empty set X a distance distribution function.

Then, for all $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between p and q is less than x.

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The **triangle inequality** for the PM space is formulated as, for all $p, q, r \in X$,

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Several approaches for (finite) **products of PM spaces** have been introduced. The **preservation** of the corresponding triangle inequality has been the crucial point in all these considerations.

(Finite) products of PM spaces

SamPla, Sempi 2010

Theorem

Consider

- *a (finite) family of PM spaces* $(X_i, \mathcal{F}_i, \tau_i)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$,
- *an n-ary operation* α *on* Δ^+ *which is increasing in each place.*

Define $\overrightarrow{\mathcal{F}}$ *on* $X := \prod_{i=1}^n X_i$, *for all* $\overrightarrow{p}$, $\overrightarrow{q}$, $\overrightarrow{r}$ *in* X, *by*

$$\overrightarrow{\mathcal{F}}(\overrightarrow{p}, \overrightarrow{q}) = \alpha(\mathcal{F}_1(p_1, q_1), \ldots, \mathcal{F}_n(p_n, q_n)).$$

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If there exists a triangle function τ *such that*

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then $\overrightarrow{\mathcal{F}}$ *satisfies the triangle inequality on* X *with respect to* τ*, so that* $(X, \overrightarrow{\mathcal{F}}, \tau)$ *is a probabilistic metric space.*

Some results

Results by Tardiff, 1976

- For continuous t-norms T_1 and T_2 the following holds:

$$\begin{aligned} T_1 \gg T_2 \quad &\Leftrightarrow \quad \pi_{T_1} \gg \pi_{T_2} \quad \Leftrightarrow \quad \tau_{T_1,+} \gg \tau_{T_2,+} \\ &\Leftrightarrow \quad \pi_{T_1} \gg \tau_{T_2,+} \quad \Leftrightarrow \quad \tau^{*}_{T_2^*,+} \gg \tau^{*}_{T_1^*,+}. \end{aligned}$$

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Constructing dominating operations from given ones

(SamPla, Sempi, 2010)

Particular case: For all triangle functions τ and all $n \in \mathbb{N}$, $n \geq 2$, it holds that

$$\tau^n \gg \tau.$$

Some results

relating to triangle functions of the type "splitting the argument"

Conditions on the "splitting the argument"

Consider two commutative and associative functions

$$L_1, L_2 \colon \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$$

such that

- both have full range,

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Assume that both are additionally

(LS) jointly strictly increasing, i.e., that for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}^+$

$$u_1 < u_2, v_1 < v_2 \quad \Rightarrow \quad L_i(u_1, v_1) < L_i(u_2, v_2),$$

(L0) and having 0 as their common neutral element.

Some results

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Proposition (SamPla, Sempi 2010)

Consider

- two left-continuous t-norms T_1, T_2, and
- two functions L_1, L_2 of the type mentioned above.

Then the following holds:

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but also

$$\pi_{T_1} \gg \tau_{T_2,L} \quad \Leftrightarrow \quad T_1 \gg T_2.$$

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Proposition **(SamPla, Sempi 2010)**

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Then we know

$$S_1^* \gg S_2^* \quad \text{and} \quad L_1 \ll L_2 \quad \Rightarrow \quad \tau^*_{S_1^*,L_1} \gg \tau^*_{S_2^*,L_2}.$$

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$$S_1^* \gg S_2^* \quad \text{and} \quad L_1 \ll L_2 \quad \Rightarrow \quad \tau^*_{S_1^*,L_1} \gg \tau^*_{S_2^*,L_2}.$$

We still do not know whether the converse is also true or not.

$$S_1^* \gg S_2^* \quad \text{and/or} \quad L_1 \ll L_2 \quad \overset{?}{\Leftarrow} \quad \tau^*_{S_1^*,L_1} \gg \tau^*_{S_2^*,L_2}.$$

Thank you for your attention

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