Triangle Functions
Constructions and Functional (In)equalities

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Motivation and interpretation
Distance distribution functions

A distribution function is a function $F: \mathbb{R} \rightarrow [0, 1]$ such that:

- $F$ is increasing;
- $F$ is left-continuous on $\mathbb{R}$;
- $F(-\infty) = 0$ and $F(\infty) = 1$.

The set of all distributions functions (d.f.) will be denoted by $\Delta$.

A distance distribution function is a distribution function $F$ such that $F(0) = 0$.

The set of all distance distribution functions (d.d.f.) will be denoted by $\Delta^+$.
Possible interpretations of elements of $\Delta^+$

- In **probabilistic metric spaces** a function $\mathcal{F}$ assigns to each pair of elements $p$ and $q$ of some non-empty set $X$ a distance distribution function.

Then, for all $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between $p$ and $q$ is less than $x$. 
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- A **finite fuzzy real** is a fuzzy subset $A$ of $\mathbb{R}$ such that
  - $A$ is increasing;
  - $A$ is left-continuous (on $\mathbb{R}$);
  - $\inf\{A(x) \mid x \in \mathbb{R}\} = 0$ and $\sup\{A(x) \mid x \in \mathbb{R}\} = 1$.

  A **non-negative finite fuzzy real** is a finite fuzzy real $A$ such that $A(0) = 0$. 

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- Interpretation of distribution functions of type $1 - F$.
- Interpretation of $\Delta^+$ as a **bounded lattice**.
The lattice structure of $\Delta^+$

**Definition**

- For all $F, G \in \Delta^+$:

  $$F \leq G :\iff \forall x \in \mathbb{R} : F(x) \leq G(x).$$
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- Top and bottom element:

  $$\varepsilon_0(x) := \begin{cases} 0, & x = 0, \\ 1, & x > 0, \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) := \begin{cases} 0, & x < \infty, \\ 1, & x = \infty, \end{cases}$$
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\end{cases} \]

\[ \implies (\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0) \text{ is a bounded lattice.} \]
Operations on $\Delta^+$

Remarks

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  \[ \rightarrow \text{Enriching the poset } (\Delta^+, \leq) \]
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Remarks

• Operations on $\Delta^+$:
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  --→ Closedness w.r.t. operations induced by the extension principle, i.e.
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Remarks

- Operations on $\Delta^+$:
  - $\rightarrow$ Enriching the poset $(\Delta^+, \leq)$ in order to achieve **algebraic structures**
  - $\rightarrow$ **Triangle functions**

- Operations on finite fuzzy reals:
  - $\rightarrow$ Closedness w.r.t. operations induced by the **extension principle**, i.e.

\[
A \boxtimes B(x) = \sup\{T(A(u), B(v)) \mid u, v \in \mathbb{R}, u \ast v = z\}
\]

with $T$ some t-norm and $\ast$ a binary operation on $\mathbb{R}$. 
**Triangle functions**

(Šerstnev, 1964)

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**Definition**

A binary operation \( \tau : \Delta^+ \times \Delta^+ \rightarrow \Delta^+ \) which fulfills for all \( F, G, H \in \Delta^+ \)

(i) \( \tau(F, \varepsilon_0) = F \),

(ii) \( \tau(F, G) \succeq \tau(F, H) \) whenever \( G \succeq H \),

(iii) \( \tau(F, G) = \tau(G, F) \),

(iv) \( \tau(F, \tau(G, H)) = \tau(\tau(F, G), H) \).

is called a **triangle function**.
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$$(\Delta^+, \tau, \leq)$$

is a commutative, partially ordered semigroup with neutral element $\varepsilon_0$$
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is called a triangle function.

$\tau$ is a triangular norm on the bounded lattice $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$.
Examples of triangle functions

- Consider some left-continuous t-norm $T$.

\[ \tau_T(F, G)(x) = \sup \{ T(F(u), G(v)) \mid u + v = x \} \]
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- Convolution
  \[ (F \ast G)(x) = \int_{[0,x]} F(x - t)dG(t) \]
Examples of triangle functions

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- Convolution
  \[
  (F \ast G)(x) = \int_{[0, x]} F(x - t) dG(t)
  \]

- Consider some ordinal sum \( C \) of product summands only.
  \[
  \sigma_C(F, G)(x) = \int_{\{ (u, v) \mid u + v < x \}} dC(F(u), G(v))
  \]
How to construct triangle functions?
Approach 1

Are there approaches similar to the construction/representation of (continuous) t-norms on \([0, 1]\)?
Triangle functions and t-norms

For some left-continuous t-norm $T$ on $[0, 1]$, $\pi_T(F, G)(x) = T(F(x), G(x))$ is a triangle function.
**Triangle functions and t-norms**

For some left-continuous t-norm $T$ on $[0, 1]$, 

$$\pi_T(F, G)(x) = T(F(x), G(x))$$

is a triangle function.

**Constructions of t-norms on $[0, 1]$**

- Ordinal sums and extensions
- Additive generators
Triangle Functions
S.Saminger-Platz

Motivation
The lattice of distance distribution functions

Constructions
Concepts known from triangular norms?
Extensions of t-norms on bounded lattices
Other constructions

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Dominance
... of t-norms
... of triangle functions

T-norms on $[0, 1]$

Triangle functions and t-norms
For some left-continuous t-norm $T$ on $[0, 1],$

$$\pi_T(F, G)(x) = T(F(x), G(x))$$

is a triangle function.

Constructions of t-norms on $[0, 1]$

- Ordinal sums and extensions
- Additive generators

$\rightarrow$ similar approaches for triangle functions?
Problem 7.9.1

"... In particular determine all continuous triangle functions and, if possible, find a representation corresponding to the one given in Theorems 5.3.8 and 5.4.1."

Theorem 5.3.8: Representation as minimum, Archimedean t-norm, or ordinal sum thereof;
Theorem 5.4.1: Representation by generators;

Problem 7.9.5

"Suppose that $T$ is a continuous t-norm. ... In particular, if $T$ is an ordinal sum is $\tau_{T,L}$ an ordinal sum?"
Approach 2
What can we infer from the construction methods for t-norms on lattices?
Consider

- a bounded lattice $(L, \leq, 0_L, 1_L)$,
- a bounded (and complete) sublattice $(S, \leq, a, b)$, and
- a t-norm $T^S: S^2 \to S$ on $S$. 

(Extensions of t-norms on bounded lattices)
Extensions of t-norms on bounded lattices
(Sam,Kle,Mes 2008)

Consider

- a bounded lattice \((L, \leq, 0_L, 1_L)\),
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Weakest and strongest extension

Determine operations \(T^L, \overline{T}^L : L^2 \to L\) such that

- \(T^L|_{S^2} = \overline{T}^L|_{S^2} = T^S\);
- for all t-norms \(T : L^2 \to L\) with \(T|_{S^2} = T^S\):

\[
T^L \leq T \leq \overline{T}^L;
\]
Consider

- a bounded lattice \((L, \leq, 0_L, 1_L)\),
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  \[
  \underline{T}^L \leq T \leq \overline{T}^L;
  \]

- \(\underline{T}^L\) resp. \(\overline{T}^L\) is a t-norm on \(L\).
Interesting sublattices of $\Delta^+$

- Step functions $\varepsilon_a := 1_{[a, \infty]}$ for all $a \geq 0$;
  $$E^+ = \{\varepsilon_a \mid a \geq 0\};$$
  $$\varepsilon_a \geq \varepsilon_b \iff a \leq b.$$  
  $$(E^+, \leq, \varepsilon_\infty, \varepsilon_0)$$
Interesting sublattices of $\Delta^+$

- **Step functions** $\varepsilon_a := 1_{[a, \infty]}$ for all $a \geq 0$:
  \[ E^+ = \{ \varepsilon_a \mid a \geq 0 \}; \]
  \[ \varepsilon_a \geq \varepsilon_b \iff a \leq b. \]
  \[ (E^+, \leq, \varepsilon_\infty, \varepsilon_0) \]

- **“Constant functions”**
  \[ \delta_{a,b} := b\varepsilon_a + (1 - b)\varepsilon_\infty \text{ for all } a \geq 0, b \in [0, 1]; \]
  \[ \Delta^+_\delta = \{ \delta_{a,b} \mid a \geq 0, b \in [0, 1] \}; \]
  \[ \delta_{s,t} \leq \delta_{s,u} \iff t \leq u \iff \delta_{t,s} \geq \delta_{u,s}. \]
  \[ (\Delta^+_\delta, \leq, \delta_{a,0}, \delta_{1,0}) \]
Strongest extension

The model

Consider

- a bounded lattice \((L, \leq, 0_L, 1_L)\),
- a bounded (and complete) sublattice \((S, \leq, a, b)\), and
- a t-norm \(T^S : S^2 \to S\) on \(S\).
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- a bounded (and complete) sublattice \((S, \leq, a, b)\), and
- a t-norm \(T^S: S^2 \rightarrow S\) on \(S\).

Define \(T: L^2 \rightarrow L\), for all \(x, y \in L\) by

\[
T(x, y) = \begin{cases} 
T^S(x, y), & \text{if } (x, y) \in S^2, \\
x \land y, & \text{otherwise}
\end{cases}
\]

If \(T\) is a t-norm, then \(T = \overline{T}_{T^S}\).
Strongest extension

The model

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- a bounded lattice \((L, \leq, 0_L, 1_L)\),
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If \(T\) is a t-norm, then \(T = \overline{T^L_T^S}\).

The particular case \((\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)\)

For any non-trivial bounded sublattice \((S, \leq, \varepsilon_\infty, \varepsilon_0)\), \(T\) is not a t-norm for some t-norm \(T^S\) on \(S\).
Triangle Functions

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The model

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Define

- \(T^{S\cup\{0,1\}} : (S \cup \{0, 1\})^2 \to (S \cup \{0, 1\})\) by

\[
T^{S\cup\{0,1\}}(x, y) := \begin{cases} 
    x \land y, & \text{if } 1 \in \{x, y\}, \\
    0, & \text{if } 0 \in \{x, y\}, \\
    T(x, y), & \text{if } (x, y) \in S^2.
\end{cases}
\]

- \(T^L_{T^S} : L^2 \to L\) by

\[
T^L_{T^S} := \begin{cases} 
    x \land y, & \text{if } 1 \in \{x, y\}, \\
    T^{S\cup\{0,1\}}(x^*, y^*), & \text{otherwise},
\end{cases}
\]

with \(x^* = \sup\{z \mid z \leq x, z \in S \cup \{0, 1\}\}\),
Weakest extension

The model

Consider

- a bounded lattice \((L, \leq, 0_L, 1_L)\),
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- \(\overline{T^L}_{T^S} : L^2 \to L\) by
  \[
  \overline{T^L}_{T^S} := \begin{cases} 
  x \land y, & \text{if } 1 \in \{x, y\}, \\
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  \end{cases}
  \]
  with \(x^* = \sup\{z \mid z \leq x, z \in S \cup \{0, 1\}\}\),

then \(\overline{T^L}_{T^S}\) is a t-norm and the smallest possible extension of \(T^S\).
Weakest extension

The particular case $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$

For $L = (\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$, each bounded and complete sublattice is appropriate for the weakest extension.
Approach 3
Look for other strategies!
Historic examples of triangle functions

- Consider some left-continuous t-norm $T$.

$$\tau_T(F, G)(x) = \sup\{ T(F(u), G(v)) \mid u + v = x \}$$

- Convolution

$$ (F * G)(x) = \int_{[0,x]} F(x - t) dG(t) $$

- Consider some ordinal sum $C$ of product summands only.

$$\sigma_C(F, G)(x) = \int_{\{(u,v) \mid u + v < x\}} dC(F(u), G(v))$$

- Consider some left-continuous t-norm $T$.

$$\pi_T(F, G)(x) = T(F(x), G(x))$$
Types of triangle functions

General remarks

Consider a binary operation on $\Delta^+$, i.e.,

$$\Theta : \Delta^+ \times \Delta^+ \to \Delta^+, \quad (F, G) \mapsto \Theta(F, G).$$

*What shall/can we do to determine the value $\Theta(F, G)(x)$?*

• Strategy 1: Pointwise induced triangle functions;
• Strategy 2: “Splitting the argument”
  • involving semicopulas;
  • involving co-semicopulas;
  • involving quasi-copulas;
• Strategy 3: “Involving measures and integrals”.

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*What shall/can we do to determine the value $\Theta(F, G)(x)$?*

Strategy 1: “Passing through the argument”

- Consider some binary function $A$ on $[0, 1]$;
- Determine $\Theta(F, G)(x)$ by

$$\Theta(F, G)(x) = A(F(x), G(x)).$$

$\rightarrow$ Pointwise induced operations $\pi_A$
Pointwise induced operations

**Theorem**

Consider a function $A: [0, 1]^2 \to [0, 1]$.

$\pi_A$ is a binary operation on $\Delta^+$.

$\iff$

$A$ is a left–continuous binary aggregation operator.
Pointwise induced operations

**Theorem**

Consider a function \( A : [0, 1]^2 \rightarrow [0, 1]. \)

\( \pi_A \) is a binary operation on \( \Delta^+ \).

\( \iff \)

\( A \) is a left–continuous binary aggregation operator.

**Theorem**

Consider a function \( T : [0, 1]^2 \rightarrow [0, 1]. \)

\( \pi_T \) is a triangle function.

\( \iff \)

\( T \) is a left–continuous t-norm.
“Splitting the argument”

<table>
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<th>General remarks</th>
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<tbody>
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Strategy 2: “Splitting the argument”

- Consider
  - some binary function $A$ on $[0, 1]$, 
  - some binary operation $L$ on $\mathbb{R}^+$, 
  - choose $\Omega = \sup$ or $\Omega = \inf$. 
- Determine $\Theta(F, G)(x)$ by
  $$\Theta_{A, L, \Omega}(x) = \Omega\{A(F(u), G(v)) \mid L(u, v) = x\}$$
“Splitting the argument”

General remarks

Consider a binary operation on $\Delta^+$, i.e.,

$$\Theta : \Delta^+ \times \Delta^+ \rightarrow \Delta^+, \quad (F, G) \mapsto \Theta(F, G).$$

*What shall/can we do to determine the value $\Theta(F, G)(x)$?*

Strategy 2: “Splitting the argument”

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  - some binary operation $L$ on $\mathbb{R}^+$,
  - choose $\Omega = \sup$ or $\Omega = \inf$.

- Determine $\Theta(F, G)(x)$ by

$$\Theta_{A, L, \Omega}(x) = \Omega\{A(F(u), G(v)) \mid L(u, v) = x\}$$

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Operations and triangle functions of the type

$$\tau_{f, L}(\tau_T, L), \quad \tau_{S^*, L}(\tau_{T^*}, L), \quad \rho_{Q, L}(\rho_T, L)$$
Operations on $\mathbb{R}^+$

The class $\mathcal{L}$

We consider binary operations $L$ on $\mathbb{R}^+$ such that

- $L$ is surjective, i.e., $\text{Ran}_L = \mathbb{R}^+$,
- $L$ is increasing in each place,
- $L$ is continuous on $\mathbb{R}^+$ except possibly at the points $(0, \infty)$ and $(\infty, 0)$.

We denote by $\mathcal{L}$ the set of all such operations.
Operations on $\mathbb{R}^+$

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We denote by $\mathcal{L}$ the set of all such operations.

Additional properties

Consider some $L \in \mathcal{L}$:

(LS) $L$ fulfills for all $u_1, u_2, v_1, v_2 \in \mathbb{R}^+$ with $u_1 < u_2, v_1 < v_2$

$$L(u_1, v_1) < L(u_2, v_2).$$

(L0) $L$ has 0 as its neutral element.
The class $\tau_{f,L}$

**Definition**

Consider some $L \in \mathcal{L}$ and a function $f : [0, 1]^2 \to [0, 1]$.

Define $\tau_{f,L} : \Delta^+ \times \Delta^+ \to [0, 1]^{\mathbb{R}^+}$ by

$$\tau_{f,L}(F, G)(x) = \sup\{f(F(u), G(v)) \mid L(u, v) = x\}.$$
The class $\tau_{f,L}$

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**Theorem**
Assume additionally that $L$ satisfies (LS) and (L0).

- If $\tau_{T,L}$ is a triangle function, then $T$ is a t-norm.

Motivation
The lattice of distance distribution functions

Constructions
Concepts known from triangular norms?
Extensions of t-norms on bounded lattices

Other constructions

Functional (in)equalities
Cauchy’s functional equation
Dominance
... of t-norms
... of triangle functions
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Consider some $L \in \mathcal{L}$ and a function $f : [0, 1]^2 \to [0, 1]$.

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Theorem

Assume additionally that $L$ satisfies (LS) and (L0).

- If $\tau_{T,L}$ is a triangle function, then $T$ is a t-norm.
- If $L$ is commutative and associative and if $T$ is a left-continuous t-norm, then $\tau_{T,L}$ is a triangle function.
The class $\tau_{S^*,L}^*$

**Definition**

Consider some $L \in \mathcal{L}$ and a co-semicopula $S^*: [0, 1]^2 \to [0, 1]$, i.e.,

$$S(x, y) = 1 - S^*(1 - x, 1 - y)$$

is a semicopula.

Define $\tau_{S^*,L}^*$: $\Delta^+ \times \Delta^+ \to [0, 1]^{\mathbb{R}^+}$ by

$$\tau_{S^*,L}^*(F, G)(x) = \inf\{S^*(F(u), G(v)) \mid L(u, v) = x\}.$$
The class $T^*_{S^*, L}$

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Define $T^*_{S^*, L}: \Delta^+ \times \Delta^+ \rightarrow [0, 1]^{\mathbb{R}^+}$ by

$$T^*_{S^*, L}(F, G)(x) = \inf \left\{ S^*(F(u), G(v)) \mid L(u, v) = x \right\}.$$

**Theorem**

Assume additionally that $L$ is commutative, associative and satisfies (LS) and (L0).

$T^*$ is a continuous t-conorm.

$\Rightarrow$

$T^*_{T^*, L}$ is a triangle function.
The class $\sigma_{C,L}$

**General remarks**

Consider a binary operation on $\Delta^+$, i.e.,

$$\Theta : \Delta^+ \times \Delta^+ \rightarrow \Delta^+, \quad (F, G) \mapsto \Theta(F, G).$$

**What shall/can we do to determine the value $\Theta(F, G)(x)$?**

Strategy 3: “Involving measures and integrals”, e.g.

- Consider
  - some copula $C$,
  - some binary operation $L$ on $\mathbb{R}^+$.
- Determine $\Theta(F, G)(x)$ by

$$\Theta(F, G)(x) = \int_{L(x)} dC(F(u), G(v))$$

with $L(x) = \{(u, v) \mid L(u, v) < x\}$. 

Involving copulas

Definition

Consider some \( L \in \mathcal{L} \) and some copula \( C \).

Define the function \( \sigma_{C,L} : \Delta^+ \times \Delta^+ \rightarrow \Delta^+ \) by

\[
\sigma_{C,L}(F, G)(0) := 0, \quad \sigma_{C,L}(F, G)(\infty) := 1
\]

and

\[
\sigma_{C,L}(F, G)(x) := \int_{L(x)} dC(F(u), G(v))
\]

for all \( x \in ]0, +\infty[ \), where

\[
L(x) = \{(u, v) \mid u, v \in \mathbb{R}^+, L(u, v) < x\}.
\]
Neutral element

Consider some $L \in \mathcal{L}$ and some copula $C$.

$$\sigma_{C,L} \text{ has } \varepsilon_0 \text{ as its neutral element.}$$

$$\Leftrightarrow$$

$L$ fulfills (L0).
Neutral element

Consider some $L \in \mathcal{L}$ and some copula $C$.

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Commutativity

(Frank, 1975, Frank 1991)

Consider some $L \in \mathcal{L}$ and some copula $C$.

$$\sigma_{C,L} \text{ is commutative. } \implies L \text{ is commutative.}$$
Properties

Neutral element
Consider some $L \in \mathcal{L}$ and some copula $C$.

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\[ L \text{ fulfills (L0).} \]

Commutativity
(Frank, 1975, Frank 1991)
Consider some $L \in \mathcal{L}$ and some copula $C$.

\[ \sigma_{C,L} \text{ is commutative.} \Rightarrow L \text{ is commutative.} \]

\[ \sigma_{C,L} \text{ is commutative.} \Leftarrow C \text{ and } L \text{ are commutative.} \]
**Properties**

**Associativity** *(Frank, 1991)*

Consider some $L \in \mathcal{L}$ and some copula $C$.

$\sigma_{C,L}$ is associative. $\Rightarrow L$ is associative.
### Properties

**Associativity**

(Frank, 1991)

Consider some $L \in \mathcal{L}$ and some copula $C$.

$$\sigma_{C,L} \text{ is associative.} \Rightarrow L \text{ is associative.}$$

<table>
<thead>
<tr>
<th>Particular $L \in \mathcal{L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider some $L \in \mathcal{L}$ satisfying (L0) and (LS) and being commutative and associative.</td>
</tr>
<tr>
<td>• $L = \max$;</td>
</tr>
<tr>
<td>• there exists some continuous and strictly increasing function $h: \mathbb{R}<em>+ \rightarrow \mathbb{R}</em>+$ with</td>
</tr>
<tr>
<td>$L(u, v) = h^{-1}(h(u) + h(v))$.</td>
</tr>
</tbody>
</table>
Characterization

Theorem (Frank, 1991)

Consider

• some $L \in \mathcal{L}$ such that $L(u, v) = h^{-1}(h(u) + h(v))$ for some
a continuous, strictly increasing function $h: \mathbb{R}_+ \to \mathbb{R}_+$;

• some copula $C$. 

Motivation

The lattice of distance distribution functions

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Other constructions

Functional (in)equalities

Cauchy’s functional equation
Dominance
... of t-norms
... of triangle functions

S. Saminger-Platz FSTA 2014 Liptovský Ján, Jan 2014
Theorem (Frank, 1991)

Consider

- some $L \in \mathcal{L}$ such that $L(u, v) = h^{-1}(h(u) + h(v))$ for some a continuous, strictly increasing function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$;
- some copula $C$.

$\sigma_{C,L}$ is associative. \iff $C$ is a (trivial or non-trivial) ordinal sum of product t-norms.
Summary

We have discussed different strategies for constructing triangle functions.

Are there triangle functions different from the types we have listed and presented above?
Functional (in)equalities
Cauchy’s functional equation
Cauchy’s functional equation

Consider a triangle function $\tau$.

A mapping $\varphi : \Delta^+ \to \Delta^+$ is a solution of the Cauchy’s functional equation for $\tau$, if, and only if, for all $F, G \in \Delta^+$,

$$\varphi(\tau(F, G)) = \tau(\varphi(F), \varphi(G)).$$
Cauchy’s functional equation

Consider a triangle function $\tau$.

A mapping $\varphi : \Delta^+ \rightarrow \Delta^+$ is a solution of the Cauchy’s functional equation for $\tau$, if, and only if, for all $F, G \in \Delta^+$,

$$\varphi (\tau (F, G)) = \tau (\varphi (F), \varphi (G)).$$

Results for triangle functions achieved by T. Riedel based on work by R.C. Powers.
Cauchy’s functional equation

Necessary conditions

Let $\varphi$ be a solution of the Cauchy equation for a triangle function $\tau$. Then the following holds:

- **Idempotent elements**: For all $F \in \Delta^+$ with $\tau(F, F) = F$, it holds that

  $$\varphi(F) = \varphi(\tau(F, F)) = \tau(\varphi(F), \varphi(F)),$$

  i.e., $\varphi$ maps idempotents to idempotents.

In particular:
Cauchy’s functional equation

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  i.e., $\varphi$ maps idempotents to idempotents.
  In particular:

- **Neutral element**:
  $$\varphi(\varepsilon_0) = \varphi(\tau(\varepsilon_0, \varepsilon_0)) = \tau(\varphi(\varepsilon_0), \varphi(\varepsilon_0)),$$
  i.e., $\varphi(\varepsilon_0)$ is an idempotent element of $\tau$. 
Cauchy’s functional equation

Some solutions

Consider an arbitrary triangle function $\tau$ and denote by $\text{Id}_\tau$ its set of idempotent elements.

Then the following functions $\varphi : \Delta^+ \rightarrow \Delta^+$ are solutions of the Cauchy’s functional equation w.r.t. $\tau$:

- **Constant functions**: For all $H \in \text{Id}_\tau$, the (constant) functions $\varphi$ defined, for all $F \in \Delta^+$, by
  \[ \varphi(F) = H. \]
**Cauchy’s functional equation**

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- **Functions** $\varphi_H$ defined, for arbitrary $H \in \text{Id}_\tau$ and all $F \in \Delta^+$, by
  \[
  \varphi_H(F) = \tau(F, H).
  \]
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In particular with $H = \varepsilon_0$, $\varphi_{\varepsilon_0} = \text{id}_{\Delta^+}$. 
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  \[
  \varphi_H(F) = \tau(F, H).
  \]
  In particular with $H = \varepsilon_0$, $\varphi_{\varepsilon_0} = \text{id}_{\Delta^+}$.

- **Powers** $\tau^n$ of $\tau$, defined, for all $n \in \mathbb{N}$, $n \geq 2$, and all $F \in \Delta^+$, by
  \[
  \varphi(F) = \tau^n(F, \ldots, F) = \tau(\tau^{n-1}(F, \ldots, F), F).
  \]
  $n$ times
  $n-1$ times
**Theorem**

Consider a sup-continuous triangle function $\tau$ and a sup-continuous function $\varphi : \Delta^+ \to \Delta^+$. Then $\varphi$ is a solution of the Cauchy’s equation if and only if

$$\varphi (\tau (\delta_{a,b}, \delta_{c,d})) = \tau (\varphi (\delta_{a,b}), \varphi (\delta_{c,d})),$$

for all $a$ and $c$ in $\mathbb{R}_+$, and for all $b$ and $d$ in $[0, 1]$, where

$$\delta_{a,b}(x) := \begin{cases} 0, & x \in [0, a], \\ b, & x \in ]a, +\infty[, \\ 1, & x = +\infty. \end{cases}$$
Cauchy’s functional equation

Remarks

• Due to Tardiff, 1975:
  • If $T$ is continuous, then $\tau_T$ is sup-continuous.
  • However, not all triangle functions are sup-continuous, e.g., convolution, $\sigma_{\Pi}$ is not.
Cauchy’s functional equation

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• Characterization of solutions $\varphi$ for the case of $\tau_T$ with $T$ a strict or nilpotent t-norm by Riedel, 1991, based on the additive generators of $T$. 
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The strict case

\( \varphi \) is a solution if and only if there exists \( k, l > 0 \) such that, for all \( x \in \mathbb{R} \) and all \( F \in \Delta^+ \)

\[
\varphi(F)(x) = g^{-1}(k \cdot g(F(l \cdot x)))
\]

with \( g \) the additive generator of the strict t-norm \( T \).
Cauchy’s functional equation

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The strict case

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$$\varphi(F)(x) = g^{-1}(k \cdot g(F(l \cdot x)))$$

with $g$ the additive generator of the strict t-norm $T$.

- Generalization of the results for $\tau_{T,L}$ for some generated $L$, i.e., $L(x, y) = f^{-1}(f(x) + f(y))$ in Riedel, 1992.
Cauchy’s functional equation for triangle functions has been studied only for triangle functions of the type

\[ \tau = \pi T, \quad \tau = \tau T, \quad \tau = \tau T, L, \]

with restriction on both the t-norm \( T \) and on the function \( L \).

What are the solution when \( \tau \) belongs to a different family of triangle functions?
Function inequality: Dominance
Definition by Schweizer and Sklar

**Definition (Schweizer, Sklar, 1983)**

Consider a partially ordered set \((P, \leq)\) and two associative binary operations \(f, g\) on \(P\) with common identity \(e\).

Then \(f\) dominates \(g\) \((f \gg g)\) if, for all \(x, y, u, v\) in \(P\),

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f(g(x, y), g(u, v)) \geq g(f(x, u), f(y, v)).
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\[
f(g(x, y), g(u, v)) \geq g(f(x, u), f(y, v)).
\]

**Remarks:**

- Due to the common neutral element, dominance of \(f\) over \(g\) induces that \(f \geq g\).
- Associativity and commutativity of an operation \(f\) ensures its self-dominance (bisymmetry).
- Dominance does not constitute a transitive relation on the set of all triangle functions.
The (hi)story of dominance of t-norms
## Known results on families of t-norms until 2005

### Family of t-norms

**Schweizer-Sklar** \((T^{SS}_\lambda)\) for \(\lambda \in [-\infty, \infty]\)  
(Sherwood, 1984)

**Aczél-Alsina** \((T^{AA}_\lambda)\) for \(\lambda \in [0, \infty]\)

**Dombi** \((T^D_\lambda)\) for \(\lambda \in [0, \infty]\)

**Yager** \((T^Y_\lambda)\) for \(\lambda \in [0, \infty]\)  
(Klement, Mesiar, Pap, 2000)

**Frank** \((T^F_\lambda)\) for \(\lambda \in [0, \infty]\)

**Hamacher** \((T^H_\lambda)\) for \(\lambda \in [0, \infty]\)  
(Sarkoci, 2005)

**Mayor-Torrens** \((T^{MT}_\lambda)\) for \(\lambda \in [0, 1]\)

**Dubois-Prade** \((T^{DP}_\lambda)\) for \(\lambda \in [0, 1]\)  
(Sam, De Baets, De Meyer, 2005)

\[ T_\lambda \gg T_\mu \]

\[ \lambda \leq \mu \]

\[ \lambda = 0, \lambda = \mu, \mu = \infty \]

\[ \lambda = 0, \lambda = \mu \]

\[ \ldots \]

### Motivation

- The lattice of distance distribution functions

### Constructions

- Concepts known from triangular norms?
- Extensions of t-norms on bounded lattices
- Other constructions

### Functional (in)equalities

- Cauchy’s functional equation
- Dominance

... of t-norms  
... of triangle functions
Known results on families of t-norms until 2005

**Family of t-norms**

Schweizer-Sklar ($T_{\lambda}^{SS}$) $\lambda \in [-\infty, \infty]$
(Sherwood, 1984)

Aczél-Alsina ($T_{\lambda}^{AA}$) $\lambda \in [0, \infty]$

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\[ T_{\lambda} \gg T_{\mu} \]
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\[ \lambda \geq \mu \]

\[ \lambda = 0, \lambda = \mu, \mu = \infty \]
\[ \lambda = 0, \lambda = \mu \]

On all these families of t-norms, dominance is **transitive** and therefore an **order relation**.
Non-transitivity of dominance for continuous t-norms

Counter-example (Sarkoci, 2008)

Let \( \lambda \in [0, \frac{1}{2}] \).

\[
\begin{align*}
T_{\lambda}^{MT} &= (\langle 0, \lambda, T_L \rangle) & T_{\lambda} &= (\langle 0, \lambda, T_L \rangle, \langle \lambda, 1, T_L \rangle) & T_L
\end{align*}
\]

Then

\[
T_{\lambda}^{MT} \gg T_{\lambda}, \quad T_{\lambda} \gg T_L, \quad T_{\lambda}^{MT} \gg T_L.
\]
Non-transitivity of dominance for continuous t-norms

Counter-example

Let $\lambda \in \left[0, \frac{1}{2}\right]$.

\[
\begin{align*}
T^\text{MT}_\lambda &= (\langle 0, \lambda, T_L \rangle) & T_\lambda &= (\langle 0, \lambda, T_L \rangle, \langle \lambda, 1, T_L \rangle) & T_L
\end{align*}
\]

Then

\[
T^\text{MT}_\lambda \succ T_\lambda, \quad T_\lambda \succ T_L, \quad T^\text{MT}_\lambda \not\succ T_L.
\]

Dominance is not transitive on the class of (continuous) t-norms.
Consider two continuous Archimedean t-norms $T_1$ and $T_2$ with additive generators $t_1$ and $t_2$. If the function

$$h = t_1 \circ t_2^{(-1)} : [0, \infty) \rightarrow [0, \infty]$$

is convex on $]0, t_2(0)[$ and ... 

$h'$ exists and $h'$ is log-convex on $]0, t_2(0)[$ \quad \Rightarrow \quad h'$ exists and $h'$ is geo-convex on $]0, t_2(0)[$ 

$h$ is log-convex on $]0, t_2(0)[$ \quad \Rightarrow \quad h$ is geo-convex on $]0, t_2(0)[$ 

$h$ fulfills gen. Mulholland 

$T_1 \gg T_2$
Easy-to-check-conditions \((T_1 \gg T_2)\) (SamPla, De Baets, De Meyer, 2009)

**Conditions on** \(h\)

Consider two continuous Archimedean t-norms \(T_1\) and \(T_2\) with sufficiently often differentiable additive generators \(t_1\) and \(t_2\). Define the function \(h = t_1 \circ t_2^{(-1)} : [0, \infty] \rightarrow [0, \infty]\). Then

- \(h\) is **convex** on \(]0, t_2(0)[\), if and only if, for all \(u \in ]0, 1[\),
  \[
  t'_1(u)t''_2(u) - t''_1(u)t'_2(u) \geq 0.
  \]

- \(h\) is **log-convex** on \(]0, t_2(0)[\), if and only if, for all \(u \in ]0, 1[\),
  \[
  t'^2_1(u)t'_2(u) + t_1(u) (t'_1(u)t''_2(u) - t''_1(u)t'_2(u)) \geq 0.
  \]

- \(h\) is **geo-convex** on \(]0, t_2(0)[\), if and only if, for all \(u \in ]0, 1[\),
  \[
  \frac{t'^2_1(u) - t_1(u)t''_1(u)}{t_1(u)t'_1(u)} \geq \frac{t'^2_2(u) - t_2(u)t''_2(u)}{t_2(u)t'_2(u)}
  \]
New results on families of t-norms

All families are taken from the book on associative functions by Alsina, Frank, Schweizer, 2006 resp. the book on copulas by Nelsen, 2006.

<table>
<thead>
<tr>
<th>Family of t-norms</th>
<th>( T_\lambda \gg T_\mu )</th>
<th>Hasse-Diag</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_\lambda^8 ), ( \lambda \in [0, \infty] )</td>
<td>log-convexity of ( h' )</td>
<td>( \lambda \leq \mu )</td>
</tr>
<tr>
<td>( T_\lambda^{15} ), ( \lambda \in [0, \infty] )</td>
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<tr>
<td>( T_\lambda^{22} ), ( \lambda \in [0, \infty] )</td>
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</tr>
<tr>
<td>( T_\lambda^{23} ), ( \lambda \in [0, \infty] )</td>
<td>geo-convexity of ( h )</td>
<td>( \lambda \leq \mu )</td>
</tr>
<tr>
<td>( T_\lambda^9 ), ( \lambda \in [0, \infty] )</td>
<td></td>
<td>( \lambda = \infty, \lambda = \mu, \mu = 0 )</td>
</tr>
</tbody>
</table>

On all these families of t-norms, dominance is transitive and therefore an order relation.
The family of Sugeno-Weber t-norms

The family of Sugeno-Weber t-norms \( (T^{SW}_\lambda)^{\lambda \in [0, \infty]} \) is given by

\[
T^{SW}_\lambda(u, v) = \begin{cases} 
T_P(u, v), & \text{if } \lambda = 0, \\
T_D(u, v), & \text{if } \lambda = \infty, \\
\max(0, (1 - \lambda)uv + \lambda(u + v - 1)), & \text{if } \lambda \in ]0, \infty[. 
\end{cases}
\]
The family of Sugeno-Weber t-norms

The family of Sugeno-Weber t-norms \( T^\text{SW}_\lambda \) \( \lambda \in [0, \infty] \) is given by

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\max(0, (1 - \lambda)uv + \lambda(u + v - 1)), & \text{if } \lambda \in ]0, \infty[. 
\end{cases}
\]

Results on dominance

- Partial results based on the differential sufficient conditions
- Full characterization now available (proven by CAD)
The family of Sugeno-Weber t-norms (Kauers, Pillwein, SamPla, 2011)

**Theorem**

Consider the family of Sugeno-Weber t-norms \((T^{SW}_\lambda)_{\lambda \in [0,\infty]}\).

\(T^{SW}_\lambda\) dominates \(T^{SW}_\mu\) if and only if one of the following holds:

(i) \(\lambda = 0\),
(ii) \(\mu = \infty\),
(iii) \(\lambda = \mu\),
(iv) \(0 < \lambda < \mu \leq 17 + 12\sqrt{2}\),
(v) \(17 + 12\sqrt{2} < \mu\) and \(0 < \lambda \leq \left(\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}}\right)^2\).
Dominance for triangle functions
Why dominance for functions on $\Delta^+$?

In probabilistic metric (PM) spaces a function $\mathcal{F}$ assigns to each pair of elements $p$ and $q$ in a non-empty set $X$ a distance distribution function.

Then, for all $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between $p$ and $q$ is less than $x$. 
Why dominance for functions on $\Delta^+$?

In probabilistic metric (PM) spaces a function $\mathcal{F}$ assigns to each pair of elements $p$ and $q$ in a non-empty set $X$ a distance distribution function.

Then, for all $x > 0$, the value $\mathcal{F}(p, q)(x)$ is interpreted as the probability that the distance between $p$ and $q$ is less than $x$.

The **triangle inequality** for the PM space is formulated as, for all $p, q, r \in X$,

$$\mathcal{F}(p, r) \geq \tau(\mathcal{F}(p, q), \mathcal{F}(p, q))$$

shall hold, where $\tau$ denotes the triangle function associated with the given PM space.
Why dominance for functions on \( \Delta^+ \)?

In probabilistic metric (PM) spaces a function \( \mathcal{F} \) assigns to each pair of elements \( p \) and \( q \) in a non-empty set \( X \) a distance distribution function.

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The **triangle inequality** for the PM space is formulated as, for all \( p, q, r \in X \),

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\mathcal{F}(p, r) \geq \tau(\mathcal{F}(p, q), \mathcal{F}(p, q))
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shall hold, where \( \tau \) denotes the triangle function associated with the given PM space.

Several approaches for (finite) **products of PM spaces** have been introduced. The **preservation** of the corresponding triangle inequality has been the crucial point in all these considerations.
Theorem

Consider

- a (finite) family of PM spaces \((X_i, \mathcal{F}_i, \tau_i), i = 1, \ldots, n, n \in \mathbb{N}\),
- an \(n\)-ary operation \(\alpha\) on \(\Delta^+\) which is increasing in each place.

Define \(\overrightarrow{\mathcal{F}}\) on \(X := \prod_{i=1}^{n} X_i\), for all \(\overrightarrow{p}, \overrightarrow{q}, \overrightarrow{r}\) in \(X\), by

\[
\overrightarrow{\mathcal{F}}(\overrightarrow{p}, \overrightarrow{q}) = \alpha(\mathcal{F}_1(p_1, q_1), \ldots, \mathcal{F}_n(p_n, q_n)).
\]
(Finite) products of PM spaces

Consider

- a (finite) family of PM spaces \((X_i, \mathcal{F}_i, \tau_i), i = 1, \ldots, n, \ n \in \mathbb{N}\),
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\]

If there exists a triangle function \(\tau\) such that

- \(\alpha\) dominates \(\tau\), \(\tau \ll \alpha\), and
- \(\tau \leq \tau_i\) for every \(i = 1, \ldots, n\),
Theorem

Consider

- a (finite) family of PM spaces \((X_i, \mathcal{F}_i, \tau_i), i = 1, \ldots, n, n \in \mathbb{N},\)

- an \(n\)-ary operation \(\alpha\) on \(\Delta^+\) which is increasing in each place.

Define \(\overrightarrow{\mathcal{F}}\) on \(X := \prod_{i=1}^{n} X_i,\) for all \(\overrightarrow{p}, \overrightarrow{q}, \overrightarrow{r}\) in \(X,\) by

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If there exists a triangle function \(\tau\) such that

- \(\alpha\) dominates \(\tau, \tau \ll \alpha,\) and

- \(\tau \leq \tau_i\) for every \(i = 1, \ldots, n,\)

then \(\overrightarrow{\mathcal{F}}\) satisfies the triangle inequality on \(X\) with respect to \(\tau,\) so that \((X, \overrightarrow{\mathcal{F}}, \tau)\) is a probabilistic metric space.
Some results

### Results by Tardiff, 1976

- For continuous t-norms $T_1$ and $T_2$ the following holds:

\[
T_1 \gg T_2 \iff \pi T_1 \gg \pi T_2 \iff \tau T_1^+, \gg \tau T_2^+, \\
\iff \pi T_1 \gg \tau T_2^+, \iff \tau T_2^*, + \gg \tau T_1^*, +.
\]
Some results

Results by Tardiff, 1976

• For continuous t-norms $T_1$ and $T_2$ the following holds:

$$T_1 \gg T_2 \iff \pi T_1 \gg \pi T_2 \iff \tau T_1, + \gg \tau T_2, +$$

$$\iff \pi T_1 \gg \tau T_2, + \iff \tau^{*} T_2, + \gg \tau^{*} T_1, + .$$

• For all triangle functions $\tau$ it holds that

$$\tau \gg \tau \text{ and } \pi M \gg \tau.$$
Some results

Results by Tardiff, 1976

- For continuous t-norms $T_1$ and $T_2$ the following holds:

\[ T_1 \gg T_2 \iff \pi T_1 \gg \pi T_2 \iff \tau T_1, + \gg \tau T_2, + \]
\[ \iff \pi T_1 \gg \tau T_2, + \iff \tau T^*_1, + \gg \tau T^*_2, +. \]

- For all triangle functions $\tau$ it holds that

\[ \tau \gg \tau \text{ and } \pi M \gg \tau. \]

Constructing dominating operations from given ones

(SamPla, Sempi, 2010)

Particular case: For all triangle functions $\tau$ and all $n \in \mathbb{N}$, $n \geq 2$, it holds that

\[ \tau^n \gg \tau. \]
Some results
relating to triangle functions of the type “splitting the argument”

Conditions on the “splitting the argument”

Consider two commutative and associative functions

\[ L_1, L_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \]

such that

- both have full range,
Some results
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- are increasing in each place, and
- are continuous on \( \bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+ \), except possibly at the points \((0, \infty)\) and \((\infty, 0)\).

Assume that both are additionally

\( (\text{LS}) \) jointly strictly increasing, i.e., that for all \( u_1, u_2, v_1, v_2 \in \bar{\mathbb{R}}^+ \)

\[ u_1 < u_2, v_1 < v_2 \Rightarrow L_i(u_1, v_1) < L_i(u_2, v_2), \]

\( (\text{L0}) \) and having 0 as their common neutral element.
Some results
relating to triangle functions of the type “splitting the argument”

<table>
<thead>
<tr>
<th>Proposition</th>
<th>(SamPla, Sempi 2010)</th>
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<tr>
<td>Consider</td>
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<tr>
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Some results
relating to triangle functions of the type “splitting the argument”

Proposition (SamPla, Sempi 2010)

Consider
- two left-continuous t-norms $T_1$, $T_2$, and
- two functions $L_1$, $L_2$ of the type mentioned above.

Then the following holds:

$$T_1 \succ T_2 \quad \text{and} \quad L_1 \prec L_2 \quad \Rightarrow \quad \tau_{T_1,L_1} \succ \tau_{T_2,L_2},$$

$$T_1 \succ T_2 \quad \Leftarrow \quad \tau_{T_1,L_1} \succ \tau_{T_2,L_2}.$$ 

Special case: $L_1 = L_2 = L$

In particular, if $L_1 = L_2 = L$, then

$$\tau_{T_1,L} \succ \tau_{T_2,L} \quad \Leftarrow \quad T_1 \succ T_2.$$
Some results relating to triangle functions of the type “splitting the argument”

Proposition (SamPla, Sempi 2010)

Consider

- two left-continuous t-norms \( T_1, T_2 \), and
- two functions \( L_1, L_2 \) of the type mentioned above.

Then the following holds:

\[
T_1 \gg T_2 \quad \text{and} \quad L_1 \ll L_2 \quad \Rightarrow \quad \tau_{T_1,L_1} \gg \tau_{T_2,L_2},
\]

\[
T_1 \gg T_2 \quad \iff \quad \tau_{T_1,L_1} \gg \tau_{T_2,L_2}.
\]

Special case: \( L_1 = L_2 = L \)

In particular, if \( L_1 = L_2 = L \), then

\[
\tau_{T_1,L} \gg \tau_{T_2,L} \quad \iff \quad T_1 \gg T_2.
\]

but also

\[
\pi_{T_1} \gg \tau_{T_2,L} \quad \iff \quad T_1 \gg T_2.
\]
Some results
relating to triangle functions of the type “splitting the argument”

Proposition

Consider

- two continuous co-semicopulas $S_1^*$, $S_2^*$,

(SamPla, Sempi 2010)
Some results
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Proposition

Consider

- two continuous co-semicopulas $S_1^*$, $S_2^*$,
e.g., they might be two continuous t-conorms, and
Some results
relating to triangle functions of the type “splitting the argument”

Proposition
(SamPla, Sempi 2010)

Consider

- two continuous co-semicopulas $S_1^*$, $S_2^*$, 
e.g., they might be two continuous t-conorms, and
- two functions $L_1$, $L_2$ of the type mentioned above.

Then we know

$$S_1^* \succ S_2^* \quad \text{and} \quad L_1 \preccurlyeq L_2 \quad \Rightarrow \quad \tau_{S_1^*,L_1} \succ \tau_{S_2^*,L_2}.$$
Some results
relating to triangle functions of the type “splitting the argument”

Proposition (SamPla, Sempi 2010)

Consider

- two continuous co-semicopulas $S_1^*$, $S_2^*$, e.g., they might be two continuous t-conorms, and
- two functions $L_1$, $L_2$ of the type mentioned above.

Then we know

$$S_1^* \gg S_2^* \quad \text{and} \quad L_1 \ll L_2 \quad \Rightarrow \quad \tau_{S_1^*, L_1} \gg \tau_{S_2^*, L_2}.$$ 

We still do not know whether the converse is also true or not.

$$S_1^* \gg S_2^* \quad \text{and/or} \quad L_1 \ll L_2 \quad \Leftarrow \quad \tau_{S_1^*, L_1} \gg \tau_{S_2^*, L_2}.$$
Thank you for your attention
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