AGGREGATION FUNCTIONS in SOCIAL NETWORKS: APPLICATION to INFLUENCE and CONTAGION

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- Examples on large scale networks: contagion, diffusion of innovation
- In many models, the opinion of an agent results from the aggregation of the opinion of the others
- **Questions:** to which opinion does each agent converge? When do we reach consensus? When do subgroups of different opinion form? Is cycling possible?

Outline

1. A model of influence based on aggregation functions

2. Anonymous influence

3. Contagion

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  - opinion changes are not deterministic

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- It is well known that the *qualitative description of convergence* (terminal classes) needs only the knowledge of the *reduced matrix* $\widetilde{\mathbf{B}}$, where

$$\widetilde{b}_{S,T} = \begin{cases} 1, & \text{if } b_{S,T} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(equivalently represented by a directed graph)

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- **Remark:** $\widetilde{\mathbf{B}}$ is insensitive to possible correlation among agents

Influential players

Definition

Consider an influence model based on aggregation functions $\mathbf{A} = (A_1, \ldots, A_n)$.

1. Agent $j \in N$ is *yes-influential in A_i* if $A_i(1_j) > 0$.
2. Agent $j \in N$ is *no-influential in A_i* if $A_i(1_{N\setminus j}) < 1$.

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The *graph of yes-influence* is a directed graph $G^{\text{yes}}_{\mathbf{A}} = (N, E)$ whose set of nodes is N, and there is an arc (j, i) from j to i if j is yes-influential in A_i. The *graph of no-influence* $G^{\text{no}}_{\mathbf{A}}$ is defined similarly.

Influential coalitions

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A nonempty coalition $S \subseteq N$ is *yes-influential* for i if

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2. For all $S' \subset S$, $A_i(1_{S'}) = 0$.

Similarly, a coalition S is *no-influential* for i if

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Call $\mathcal{C}_i^{\mathrm{yes}}$ and $\mathcal{C}_i^{\mathrm{no}}$ the collections of yes- and no-influential coalitions for i. These are nonempty antichains in 2^N.

Influential coalitions

$n =$	2	3	4	5	6	7	8
size of the transition matrix	16	64	256	1024	4096	16384	65536
total maximal size of $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$	8	18	48	100	240	490	1120

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Conclusion: it is not necessary to know the aggregation functions for a qualitative description of convergence

Some results on qualitative convergence

Theorem

Consider an influence process $\mathbf{B}$ *based on aggregation functions. Then terminal classes are:*

1. *either singletons* $\{S\}$, $S \in 2^N$;
2. *or cycles of nonempty sets* $\{S_1, \dots, S_k\}$ *of any length* $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$ *(and therefore they are periodic of period* k*) with the condition that all sets are pairwise incomparable;*
3. *or collections* $\mathcal{C}$ *of nonempty sets with the property that* $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p$, *where each subcollection* $\mathcal{C}_j$ *is a Boolean lattice* $[S_j, S_j \cup K_j]$, $S_j \neq \emptyset$, $S_j \cup K_j \neq N$, *and at least one* K_j *is nonempty.*

We call cyclic terminal classes those terminal classes of the second type and regular terminal classes those of the third type. Regular terminal classes can be periodic. Regular terminal classes formed by a single Boolean lattice are called Boolean terminal classes.

Example

Consider $N = \{1, 2, 3\}$ and the following aggregation functions:

$$
\begin{array}{lll}
A_1(1\ 0\ 0) = 1 & A_2(1\ 0\ 0) = 0.5 & A_3(1\ 0\ 0) = 0 \\
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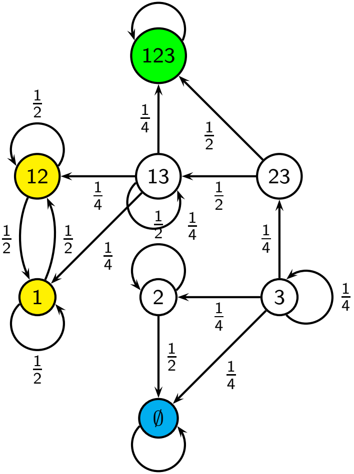
This gives the following digraph for the Markov chain:



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Proposition

1. *If the graph $(G^{\text{yes}}_{\mathbf{A}})^* \cup G^{\text{no}}_{\mathbf{A}}$ is strongly connected, then there is no nontrivial terminal state, where $(\cdot)^*$ indicates the graph with inverted arcs*
2. *If for all $i \in N$, i is yes- and no-influential for i, then there is no cyclic class*

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For an agent $i \in N$, its *closure* in $G_{\mathbf{A}}^{\mathrm{no}}$, denoted by $\mathrm{cl}(i)$, is the set of agents which can reach i by a path in $G_{\mathbf{A}}^{\mathrm{no}}$. By convention, $i \in \mathrm{cl}(i)$.

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Theorem

There is no normal regular terminal class if for each $i \in N$, every agent outside $\mathrm{cl}(i)$ can be reached by a path from $\mathrm{cl}(i)$ in $G^{\mathrm{yes}}_{\mathbf{A}}$.

Symmetric decomposable models

- An influence model based on aggregation functions $\mathbf{A}$ is *decomposable* if all influential coalitions are singletons (i.e., all can be described by influential players, hence by $G_{\mathbf{A}}^{\mathrm{yes}}$ and $G_{\mathbf{A}}^{\mathrm{no}}$)

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Theorem

Any symmetric decomposable model is qualitatively equivalent to a unique WAM (weighted arithmetic mean) model. Conversely, any WAM model is qualitatively equivalent to some symmetric decomposable model.

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- People follow anonymous customers that have expressed their positive/negative opinion on the product

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- More generally, OWA models this kind of behavior:

$$\mathrm{OWA}_W(x) = \sum_{i=1}^{n} w_i x_{(i)}$$

with $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$, and $w_i \geq 0 \ \forall i$, $\sum_i w_i = 1$.

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with $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$, and $w_i \geq 0\ \forall i$, $\sum_i w_i = 1$.
- We propose a model where each agent j has an OWA function with weight vector w^j. Recall that $\mathrm{OWA}_{w^j}(1_S)$ is the probability that agent j will say 'yes' given that the current set of 'yes'-voters is S.

Convergence in the anonymous model

The following theorem gives us a necessary and sufficient condition for convergence to consensus.

Theorem

Consider the aggregation model $A_i = \mathrm{OWA}_{w^i}, i \in N$. *Then, there are only the trivial terminal classes if and only if there exists* $\bar{k} \in \{1, \ldots, n\}$ *s.t. both:*

1. *For all* $k = \bar{k}, \ldots, n-1$, *there are* $k+1$ *distinct agents s.t. coalitions of size* k *are "yes"-influential each on them.*
2. *For all* $k = 1, \ldots, \bar{k}-1$, *there are* $n-k+1$ *distinct agents s.t. coalitions of size* $n-k$ *are "no"-influential on each of them.*

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2. *For all $k = 1, \dots, \bar{k}-1$, there are $n-k+1$ distinct agents s.t. coalitions of size $n-k$ are "no"-influential on each of them.*

- In other words, we have a cascade that leads either to the "yes"- (part (i)) or "no"-consensus (part (ii))

Generalization

- We say that an aggregation function A is OWA_w-*decomposable* if there exists $\lambda \in]0,1]$ and an aggregation function A' s.t.

$$A = \lambda \text{OWA}_w + (1-\lambda)A'.$$

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- It turns out that the **sufficiency part** of our Theorem also holds if agents use such decomposable aggregation functions.

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- Consider $n = 3$ and that whenever only two agents are of the same opinion, the third changes her opinion with probability $\lambda \in (0, 1)$:

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- This function is OWA_w-decomposable and by the above Theorem ($\bar{k} = 2$) the group eventually reaches a consensus

Fuzzy linguistic quantifiers

- Agents might adjust their opinion according to *soft majorities/minorities*, e.g., they could say "yes" if "*most* of the agents say 'yes' "
- Words like "most" or "many" are *fuzzy linguistic quantifiers*

We define a quantifier by a function of the agents' proportion saying "yes" to the degree the quantifier is satisfied.

Definition: Fuzzy linguistic quantifier

A *fuzzy linguistic quantifier* $\mathcal{Q}$ is defined by a nondecreasing function

$$\mu_{\mathcal{Q}} : [0,1] \to [0,1] \text{ s.t. } \mu_{\mathcal{Q}}(0) = 0 \text{ and } \mu_{\mathcal{Q}}(1) = 1.$$

For all quantifiers, there exists a corresponding ordered weighted average that represents it. We can find its weights as follows.

Lemma (Yager, 1988)

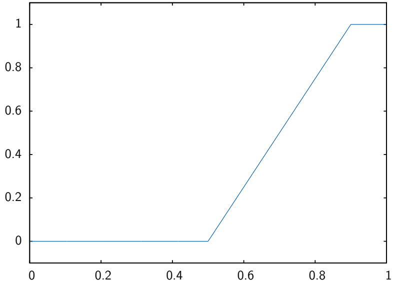
Let $\mathcal{Q}$ be a fuzzy linguistic quantifier defined by $\mu_{\mathcal{Q}}$. Then, the weights of its corresponding $\mathrm{OWA}_{\mathcal{Q}}$ are given by

$$w_k = \mu_{\mathcal{Q}}\left(\frac{k}{n}\right) - \mu_{\mathcal{Q}}\left(\frac{k-1}{n}\right), \text{ for } k = 1, \ldots, n.$$

Example: Typical quantifiers

We define

(i) $\mathcal{Q}_{\text{aa}} =$ "almost all" by

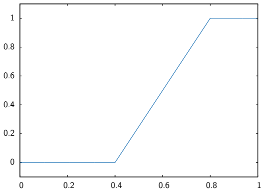


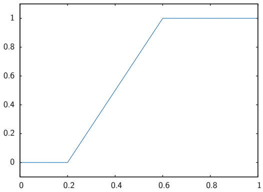
Examples

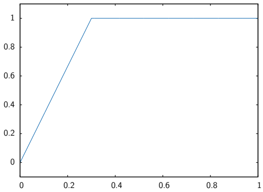
(ii) $\mathcal{Q}_{\text{mo}} =$ "most" by

(iii) $\mathcal{Q}_{\text{ma}} =$ "many" by

(iv) $\mathcal{Q}_{\text{af}} =$ "at least a few" by







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$\Rightarrow$ If agents use similar quantifiers and not too many agents deviate, they will eventually reach a consensus

Outline

1. A model of influence based on aggregation functions

2. Anonymous influence

3. Contagion

Introduction

- *Contagion* occurs if an action can spread from a finite set of individuals to the whole population

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- We show that this model is a particular instance of our influence model based on aggregation functions

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- The contagion model is a particular influence model based on the following aggregation function for each player x:
$$A_x(1_X) = \begin{cases} 1, & \text{if } \frac{|X \cap \Gamma(x)|}{\gamma} \geq q \\ 0, & \text{otherwise.} \end{cases}$$

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- If q is below the contagion threshold, the terminal states are the trivial states $\emptyset$ and $\mathcal{X}$. Otherwise, other nontrivial terminal classes can occur.

Analysis of convergence

The analysis shows that no regular terminal exists, but it may exist non trivial terminal states and cycles.

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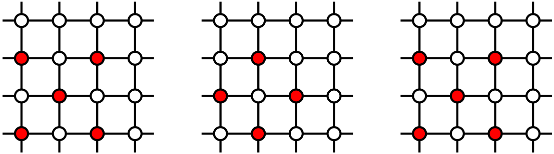
Example of a cycle: the 2-dim mesh with 4 neighbors ($q = \frac{1}{2}$)

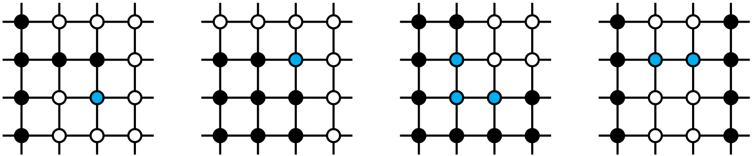


etc.

Analysis of convergence

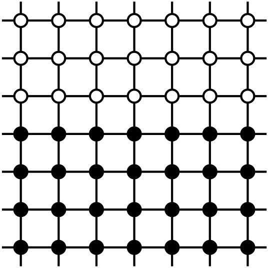
Example of nontrivial terminal state: the 2-dim mesh with 4 neighbors

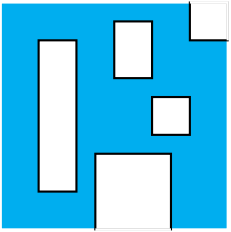
Remarkable configurations (from left to right): antenna, convex corner, concave corner, isthm



Analysis of convergence

It can be shown that only $q = \frac{1}{2}$ or $\frac{3}{4}$ lead to nontrivial terminal classes. Also, these two cases are exact complements of each other, in the sense that S is a possible terminal state for $q = \frac{1}{2}$ iff $\mathcal{X} \setminus S$ is a possible terminal state for $q = \frac{3}{4}$. Taking for example the latter, each connected component of S should be of size at least 4, and should have no convex corner, no antenna and no isthm, while each connected component of the complement set should be of size at least 3 and have no antennas.





References

Most of the present material can be found in

1. M. Grabisch and A. Rusinowska, A model of influence based on aggregation functions. Mathematical Social Sciences, Vol. 66 (2013), 316-330.
2. M. Förster, M. Grabisch and A. Rusinowska, Anonymous social influence. Games and Economic Behavior, Vol. 82 (2013), 621-635.

Thank you for your attention!