Representations of MV-algebras by Hilbert-space effects

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Introduction

Recently, it has been shown that every effect algebra possessing an ordering set of states $\mathcal{S}$ can be embedded into the effect algebra of the Hilbert space effects (i.e., operators between the zero and identity operator) on $\ell_2(\mathcal{S})$[a].

MV-algebras form a special subclass of effect algebras, and it is well known every Archimedean MV-algebra M has an ordering set of extremal states $\mathcal{S}_0$. In this contribution, we show that this enables us to prove that there is an MV-algebra embedding of M into the MV-algebra of all multiplication effects on $\ell_2(\mathcal{S}_0)$.[b]

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[a] *Riečanová and Zajac, IJTP 2010*

[b] *SP,Representations of MV-algebras by Hilbert-space effects, IJTP 2013.*

MV-algebra

An *MV-algebra* is a (2,1,0)-type algebra $(M; \boxplus, \neg, 0)$ such that $\boxplus$ and $\neg$ satisfy the identities

$$\begin{aligned}
(x \boxplus y) \boxplus z &= x \boxplus (y \boxplus z), \\
x \boxplus y &= y \boxplus x, \\
x \boxplus 0 &= x, \ \neg\neg x = x, \ x \boxplus \neg 0 = \neg 0, \\
x \boxplus \neg(x \boxplus \neg y) &= y \boxplus \neg(y \boxplus \neg x).
\end{aligned}$$

On every MV-algebra, a partial order $\leq$ is defined by the rule

$$x \leq y \ \Leftrightarrow \ y = x \boxplus \neg(x \boxplus \neg y).$$

In this partial order, every MV-algebra is a distributive lattice bounded by 0 and $\neg 0 (=: 1)$.

In any MV-algebra one can define further operations as follows:

$x \boxdot y = \neg(\neg x \boxplus \neg y)$; $x \boxminus y = x \boxdot \neg y$;

$x \vee y = \neg(\neg x \boxplus y) \boxplus y$; $x \wedge y = \neg(\neg x \vee \neg y)$.

Effect algebras

An *effect algebra* (EA) $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions[a]:

(E1) If $a \oplus b$ is defined then $b \oplus a$ is defined, and $a \oplus b = b \oplus a$ (commutativity).

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$ (orthosupplementation).

(E4) If $a \oplus 1$ is defined then $a = 0$ (zero-one law).

[a] Foulis, Bennett, 1994

Properties of EAs

In an EA $(E; \oplus, 0, 1)$ we define:

- partial order by $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$, then $0 \leq a \leq 1$ for all $a \in E$, $a \oplus b$ is defined iff $a \leq b'$;

- $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$;

- a and b are *orthogonal* ($a \perp b$) iff $a \oplus b$ is defined.

- a finite sequence of elements $a_1, a_2, \ldots, a_n$ (not necessarily all different) is *orthogonal* iff $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists (the latter $\oplus$-sum is defined by recurrence).

Examples of EAs

• $(G; \leq, +, 0)$ a partially ordered abelian group, $u \in G$, $u \geq 0$, $G[0, u] = \{x \in G : 0 \leq x \leq u\}$ can be organized into an effect algebra by defining $x \perp y$ iff $x + y \leq u$, and in this case, $x \oplus y = x + y$. Effect algebras of this type are called *interval effect algebras*.

• Let H be a (complex) Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of all bounded operators on H. Its self-adjoint part $\mathcal{B}(H)^{sa}$, forms an abelian partially ordered group.

$\mathcal{E}(H)$–self-adjoint operators in the interval between zero and identity operator are called *Hilbert space effects*.

MV-effect algebras

• An *MV-effect algebra* is a lattice ordered effect algebra with the identity

$$(a \vee b) \ominus a = b \ominus (a \wedge b).$$

There is a natural one-to-one correspondence between MV-effect algebras and MV-algebras[a]:

•Let $(M; \oplus, 0, 1)$ be an MV-effect algebra. Define the total operation $\boxplus$ as follows: $x \boxplus y = x \oplus (x' \wedge y)$. Then $(M; \boxplus, ', 0)$ is an MV-algebra.

•Conversely, let $(M; \boxplus, \neg, 0)$ be an MV-algebra. Restrict the operation $\boxplus$ to the pairs (x, y) satisfying $x \leq \neg y$ and denote the new partial operation by $\oplus$. Then $(M; \oplus, 0, \neg 0)$ is an MV-effect algebra.

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[a] Chovanec, Kôpka, 1995

Morphisms

E_1, E_2 – effect algebras, $\phi : E_1 \to E_2$ is called a *morphism of effect algebras* iff

- $\phi(1) = 1$,
- for all $a, b \in E$, if $a \oplus b$ exists in E_1, then $\phi(a) \oplus \phi(b)$ exists in E_2, and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.
- A morphism ϕ is an *isomorphism* iff ϕ is bijective and ϕ^{-1} is again a morphism.
- A morphism $\phi : E_1 \to E_2$ is an *embedding* iff $e_1 \perp e_2 \Leftrightarrow \phi(e_1) \perp \phi(e_2)$

If E_1 and E_2 are MV-effect algebras, then a morphism of effect algebras $\phi : E_1 \to E_2$ is an MV-algebra morphism iff ϕ preserves lattice operations.

States

- A *state* on an effect algebra E is a morphism $s : E \to [0,1]$, i.e., s is a mapping satisfying (S1) $s(1) = 1$, and (S2) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ is defined.

- A set $\mathcal{S}$ of states on an effect algebra E is called *ordering* iff the following holds:

$$a, b \in E, a \leq b \;\Leftrightarrow\; s(a) \leq s(b) \, \forall s \in \mathcal{S}$$

- $s_u : \mathcal{E}(H) \to [0,1]$, $s_u(a) = < u, au >, u \in H$, $\| u \| = 1$, is a (completely additive) state on $\mathcal{E}(H)$. The set of all vector states is ordering on $\mathcal{E}(H)$.

A representation of EAs

- E – effect algebra, $\mathcal{S}$ – ordering set of states on E, $\ell_2(\mathcal{S})$ – the complex Hilbert space of functions

$$x : \mathcal{S} \to \mathcal{C}, \sum_{s \in \mathcal{S}} | x(s) |^2 < \infty.$$

Define $\phi : E \to \mathcal{E}(\ell_2(\mathcal{S}))$ by

$$\phi(e)(x(s))_{s \in \mathcal{S}} = (s(e)x(s))_{s \in \mathcal{S}}, x \in \ell_2(\mathcal{S}),$$

i.e., $\phi(e)$ acts as the multiplication by the function $f_e : \mathcal{S} \to [0,1]$, $f_e(s) = s(e)$.

Then ϕ is an embedding of E into $\mathcal{E}(\ell_2(\mathcal{S}))$[a].

[a] Riečanová, Zajac, 2010

Multiplication effects

- $A_i(x_s)_{s\in\mathcal{S}} = (f_i(s)x_s)_{s\in\mathcal{S}}, i = 1, 2$ – multiplication effects on $\ell_2(\mathcal{S})$. Define
$A_1 \oplus A_2 := ((f_1 + f_2)(s)x_s)_{s\in\mathcal{S}}$ iff $f_1 + f_2 \leq 1$;
$A_1 \sqcap A_2(x_s)_{s\in\mathcal{S}} := \frac{1}{2}(A_1 + A_2 - \mid A_1 - A_2 \mid)(x_s)_{s\in\mathcal{S}}$
$= (\frac{1}{2}(f_1 + f_2 - \mid f_1 - f_2 \mid)(s)x_s)_{s\in\mathcal{S}}$
$= (\min(f_1, f_2)(s)x_s)_{s\in\mathcal{S}}$.
- The set of all multiplication effects with the operation $\oplus$ forms an effect subalgebra of $\mathcal{E}(\ell_2(\mathcal{S}))$, $A_1 \sqcap A_2$ is the infimum of A_1 and A_2 in the set of all multiplication effects[a] and it is easy to check that $A_1 \sqcup A_2 - A_1 = A_2 - A_1 \sqcap A_2$. Therefore, the set of all multiplication effects can be organized into an MV-effect algebra.

[a] Kadison, 1951

Archimedean MV-algebras

- An EA E is *Archimedean* iff for every nonzero $a \in E$ there is a maximal $n \in \mathbb{N}$ with $na \leq 1$.

- A state s on an MV-effect algebra (equivalently MV-algebra) M is extremal (in the convex set of states) iff

$$s(a \wedge b) = \min(s(a), s(b)) \; \forall a, b \in M.$$

- The following are equivalent for an MV-algebra M[a]:
(a) M has an ordering set of states;
(b) M is Archimedean;
(c) M has an ordering set of extremal sates.

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[a]Theorem 4.3, Dvurečenskij +SP, 2000

Representation of MV-algebras

M – Archimedean MV (effect) algebra,
$\mathcal{S}_0$ – ordering set of extremal states,
$\phi : M \to \mathcal{E}(\ell_2(\mathcal{S}_0))$ – representation of M.
• The range $\phi(M) := \{\phi(e) : e \in M\}$ is a subeffect algebra of $\mathcal{E}(\ell_2(\mathcal{S}_0))$ consisting of multiplication effects $\phi(e)$ corresponding to functions $f_e(s) = s(e), e \in M, s \in \mathcal{S}_0$.
• For all $e_1, e_2 \in M$ and $s \in \mathcal{S}_0$,

$$f_{e_1} \wedge f_{e_2}(s) = \min(s(e_1), s(e_2)) = s(e_1 \wedge e_2) = f_{e_1 \wedge e_2}(s),$$

which implies that
$\phi(e_1) \sqcap \phi(e_2) = \phi(e_1 \wedge e_2) \in \phi(M)$.
It follows that $\phi(M)$ is a lattice, hence an MV-algebra.

Prime states

- A state s on an EA E is *prime* iff $s(a \wedge b) = \min(s(a), s(b))$ whenever $a \wedge b$ exists in E.

E - effect algebra, S – ordering set of states, $\phi : E \to \mathcal{E}(\ell_2(S))$ – representation of E.

- $a, b \in E, a \wedge b \in E \implies \phi(a \wedge b) = \phi(a) \sqcap \phi(b)$ iff every $s \in S$ is prime.

$A, B \in \mathcal{E}(H)$, $AB = BA$, then $A \sqcap B$ is a maximal lower bound of A and B[a], hence if $A \wedge B \in \mathcal{E}(H)$, then $A \sqcap B = A \wedge B$.

- $a \wedge a' = 0 \implies s(a) \in \{0, 1\}$ for all prime states $\implies$ If E admits a representation in $\ell_2(\mathcal{S}_0)$, where $\mathcal{S}_0$ is ordering set of prime states, then $\phi(a)$ is a projection for every sharp a.

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[a] Gudder, 1996

Semiclassical EA

• An effect algebra E is *semiclassical* iff E is isomorphic to a sub-effect algebra of $[0,1]^X$.

An effect algebra E is semiclassical iff it has an ordering set of states [a].

• $\mathcal{E}(H)$ has an ordering set of states, hence is semiclassical $\implies$ (Gudder) *Effect algebras are not adequate models for QM*.

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[a] Gudder, 2010

Hidden variables

- An EA E admits *quasi hidden variables* iff E has an embedding preserving existing infima into an MV-algebra.

E admits hidden variables iff it has a generating set of prime states. [a]

- $\mathcal{E}(H)$, $\dim H \geq 3$ has no prime states, hence it does not admit quasi hidden variables.

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[a] SP, 2005

A physical system is

- *classical* – the set of events forms a Boolean algebra;
- *classical and fuzzy* – MV-algebras
- *quantum* – OMLs or OMPs
- *quantum and fuzzy* – effect algebras

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