Representation of functions between semisimple MV-algebras

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Outline

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Introduction

For MV-algebras, the so-called tense operators were already introduced by Diaconescu and Georgescu. Tense operators express the quantifiers “it is always going to be the case that” and “it has always been the case that” and hence enable us to express the dimension of time in the logic.

A crucial problem concerning tense operators is their representation. Having a MV-algebra with tense operators, Diaconescu and Georgescu asked if there exists a frame such that each of these operators can be obtained by their construction for $[0,1]$. We solve this problem for semisimple MV-algebras, i.e. those having a full set of MV-morphisms into a standard MV-algebra $[0,1]$.

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Basic definition – MV-algebras

Definition (Chang, 1958)

An ***MV-algebra*** $\mathscr{M} = (M; \oplus, \odot, \neg, 0, 1)$ is a structure where $\oplus$ is associative and commutative with neutral element 0, and, in addition, $\neg 0 = 1, \neg 1 = 0, x \oplus 1 = 1, x \odot y = \neg(\neg x \oplus \neg y)$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$ for all $x, y \in M$.

MV-algebras are a natural generalization of Boolean algebras. Namely, whilst Boolean algebras are algebraic semantics of Boolean two-valued logic, MV-algebras are algebraic semantics for Łukasiewicz many valued logic.

Example

An example of a MV-algebra is the real unit interval $[0,1]$ equipped with the operations

$$\neg x = 1 - x, x \oplus y = \min(1, x+y), x \odot y = \max(0, x+y-1)$$

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Basic definitions – MV-algebras

Every MV-algebra $\mathscr{M}$ determines a dual MV-algebra $\mathscr{M}^{op} = (M; \oplus^{op}, \odot^{op}, \neg^{op}, 0^{op}, 1^{op})$ such that $\oplus^{op} = \odot, \odot^{op} = \oplus$, $\neg^{op} = \neg, 0^{op} = 1$ and $1^{op} = 0$.

On every MV-algebra $\mathscr{M}$, a partial order $\leq$ is defined by the rule

$$x \leq y \iff \neg x \oplus y = 1.$$

In this partial order, every MV-algebra is a distributive lattice bounded by 0 and 1.

An MV-algebra is said to be *linearly ordered* (or a *MV-chain*) if the order is linear.

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Given a positive integer $n \in \mathbb{N}$, we let $nx = x \oplus x \oplus x \cdots \oplus x$, n times, $x^n = x \odot x \odot x \cdots \odot x$, n times, $0x = 0$ and $x^0 = 1$.

In every MV-algebra $\mathscr{M}$ the following equalities hold (whenever the respective join or meet exist):

(i) $a \oplus \bigvee_{i \in I} x_i = \bigvee_{i \in I}(a \oplus x_i)$, $a \oplus \bigwedge_{i \in I} x_i = \bigwedge_{i \in I}(a \oplus x_i)$,

(ii) $a \odot \bigvee_{i \in I} x_i = \bigvee_{i \in I}(a \odot x_i)$, $a \odot \bigwedge_{i \in I} x_i = \bigwedge_{i \in I}(a \odot x_i)$.

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Basic definitions – MV-morphisms and filters

Morphisms of MV-algebras (shortly *MV-morphisms*) are defined as usual, they are functions which preserve the binary operations $\oplus$ and $\odot$, the unary operation $\neg$ and nullary operations 0 and 1.

A *filter* of a MV-algebra $\mathscr{M}$ is a subset $F \subseteq M$ satisfying:
(F1) $1 \in F$
(F2) $x \in F,\ y \in M,\ x \leq y \ \Rightarrow \ y \in F$
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A filter is said to be *proper* if $0 \notin F$. Note that there is one-to-one correspondence between filters and congruences on MV-algebras.

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Basic definitions – Prime and maximal filters

A filter Q is *prime* if it satisfies the following conditions:

(P1) $0 \notin Q$.
(P2) For each x, y in M such that $x \vee y \in Q$, either $x \in Q$ or $y \in Q$.
In this case the corresponding factor MV-algebra $\mathscr{M}/Q$ is linear.

A filter U is *maximal* (and in this case it will be also called an ultrafilter) if $0 \notin U$ and for any other filter F of $\mathscr{M}$ such that $U \subseteq F$, then either $F = M$ or $F = U$. There is a one-to-one correspondence between ultrafilters and MV-morphisms from $\mathscr{M}$ into $[0,1]$.

An MV-algebra $\mathscr{M}$ is called *semisimple* if the intersection of all its maximal filters is $\{1\}$.

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Basic definitions – Boolean elements and states

An element a of a MV-algebra $\mathscr{M}$ is said to be *Boolean* if $a \oplus a = a$. We say that a MV-algebra $\mathscr{M}$ is *Boolean* if every element of $\mathscr{M}$ is Boolean. For a MV-algebra $\mathscr{M}$, the set $B(\mathscr{M})$ of all Boolean elements is a Boolean algebra.

We say that a *state* on a MV-algebra $\mathscr{M}$ is any mapping $s : M \to [0,1]$ such that (i) $s(1) = 1$, and (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0$.

A state s is *extremal* if, for all states s_1, s_2 such that $s = \lambda s_1 + (1-\lambda)s_2$ for $\lambda \in (0,1)$ we conclude $s = s_1 = s_2$.

We recall that a state s is extremal iff $\{a \in M : s(a) = 1\}$ is an ultrafilter of $\mathscr{M}$ iff $s(a \oplus b) = \min\{s(a)+s(b), 1\}$, $a, b \in M$ iff s is a morphism of MV-algebras.

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Dyadic numbers and MV-terms

The set $\mathbb{D}$ of dyadic numbers is the set of the rational numbers that can be written as a finite sum of power of 2.

If a is a number of $[0,1]$, a dyadic decomposition of a is a sequence $a^* = (a_i)_{i\in\mathbb{N}}$ of elements of $\{0,1\}$ such that $a = \sum_{i=1}^{\infty} a_i 2^{-i}$. We denote by a_i^* the i^{th} element of any sequence (of length greater than i) a^*.

If a is a dyadic number of $[0,1]$, then a admits a unique finite dyadic decomposition, called the *dyadic decomposition of* a.

If a^* is a dyadic decomposition of a real a and if k is a positive integer then we denote by $\ulcorner a^* \urcorner_k$ the finite sequence $(a_1,\ldots,a_k)$ defined by the first k elements of a^* and by $\llcorner a^* \lrcorner_k$ the dyadic number $\sum_{i=1}^{k} a_i 2^{-i}$.

Dyadic numbers and MV-terms

Definition (Ostermann,Teheux)

We denote by $f_0(x)$ and $f_1(x)$ the terms $x \oplus x$ and $x \odot x$ respectively, and by $T_{\mathbb{D}}$ the clone generated by $f_0(x)$ and $f_1(x)$.

We also denote by $g_{.}$ the mapping between the set of finite sequences of elements of $\{0,1\}$ (and thus of dyadic numbers in $[0,1]$) and $T_{\mathbb{D}}$ defined by:

$$g_{(a_1,\ldots,a_k)} = f_{a_k} \circ \cdots \circ f_{a_1}$$

for any finite sequence $(a_1,...,a_k)$ of elements of $\{0,1\}$. If $a = \sum_{i=1}^{k} a_i 2^{-i}$, we sometimes write g_a instead of $g_{(a_1,\ldots,a_k)}$.

We also denote, for a dyadic number $a \in \mathbb{D} \cap [0,1)$ and a positive integer $k \in \mathbb{N}$ such that $2^{-k} \leq 1-a$, by $l(a,k) : [a, a+2^{-k}] \to [0,1]$ a linear function defined as follows $l(a,k)(x) = 2^k(x-a)$ for all $x \in [a, a+2^{-k}]$.

MV-terms on the interval $[0,1]$

Lemma (Teheux)

If $a^* = (a_i)_{i\in\mathbb{N}}$ and $x^* = (x_i)_{i\in\mathbb{N}}$ are dyadic decompositions of two elements of $a, x \in [0,1]$, then, for any positive integer $k \in \mathbb{N}$,

$$g_{\ulcorner a^* \urcorner_k}(x) = \begin{cases} 1 & \text{if } x > \sum_{i=1}^{k} a_i 2^{-i} + 2^{-k} \\ 0 & \text{if } x < \sum_{i=1}^{k} a_i 2^{-i} \\ l(\llcorner a^* \lrcorner_k, k)(x) = \sum_{i=1}^{\infty} x_{i+k} 2^{-i} & \text{otherwise.} \end{cases}$$

Note that for any finite sequence $(a_1, \ldots, a_k)$ of elements of $\{0,1\}$ such that $a_k = 0$ we have that $g_{(a_1,\ldots,a_k)} = g_{(a_1,\ldots,a_{k-1})} \oplus g_{(a_1,\ldots,a_{k-1})}$ and clearly any dyadic number a corresponds to such a sequence $(a_1, \ldots, a_k)$.

Corollary (Teheux)

Let us have the standard MV-algebra $[0,1]$, $x \in [0,1]$ and $r \in (0,1) \cap \mathbb{D}$. Then there is a term t_r in $T_{\mathbb{D}}$ such that

$$t_r(x) = 1 \quad \text{if and only if} \quad r \le x.$$

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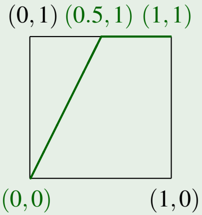
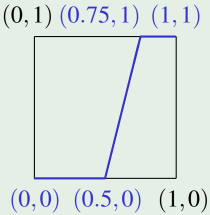
Functions t_r on unit interval $[0,1]$

Example



Function $t_{\frac{1}{2}} = g_0$



Function $t_{\frac{3}{4}} = g_{10}$

Outline

Filters, ultrafilters and the term t_r

Lemma

Let $\mathscr{M}$ be a linearly ordered MV-algebra, $s : M \to [0,1]$ an MV-morphism, $x \in M$ such that $s(x) = 1$. Then, for any $n \in \mathbb{N}, n > 1$, $n \times x = 1$.

Proposition

*Let $\mathscr{M}$ be a linearly ordered MV-algebra, $s : M \to [0,1]$ an MV-morphism, $x \in M$. Then $s(x) = 1$ iff $t_r(x) = 1$ for all $r \in (0,1) \cap \mathbb{D}$.
Equivalently, $s(x) < 1$ iff there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $t_r(x) \neq 1$. In this case, $s(x) < r$.*

Filters, ultrafilters and the term t_r

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Filters, ultrafilters and the term t_r

Proposition

Let $\mathscr{M}$ be an MV-algebra, $x \in M$ and F be any filter of $\mathscr{M}$. Then there is an MV-morphism $s : M \to [0,1]$ such that $s(F) \subseteq \{1\}$ and $s(x) < 1$ if and only if there is a dyadic number $r \in (0,1) \cap \mathbb{D}$ such that $t_r(x) \notin F$.

Corollary

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Filters, ultrafilters and the term t_r

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Outline

Semistates on MV-algebras

In this section we characterize arbitrary meets of MV-morphism into a unit interval as so-called semi-states.

Definition

Let $\mathbf{A}$ be an MV-algebra. A map $s : \mathbf{A} \to [0,1]$ is called *a semi-state on* $\mathbf{A}$ if

(i) $s(1) = 1$,

(ii) $x \leq y$ implies $s(x) \leq s(y)$,

(iii) $s(x) = 1$ and $s(y) = 1$ implies $s(x \odot y) = 1$,

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Strong semistates on MV-algebras

Definition

Let $\mathbf{A}$ be an MV-algebra. A map $s : \mathbf{A} \to [0,1]$ is called *a strong semi-state on* $\mathbf{A}$ if it is a semistate such that

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(viii) $s(x \wedge y) = s(x) \wedge s(y)$,

(ix) $s(x^n) = s(x)^n$ for all $n \in \mathbb{N}$,

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Meets of MV-morphism

Lemma

Let $\mathbf{A}$ be an MV-algebra, S a non-empty set of semi-states (strong semi-states) on $\mathbf{A}$. Then the point-wise meet $t = \bigwedge S : \mathbf{A} \to [0,1]$ is a semi-state (strong semi-state) on $\mathbf{A}$.

Lemma

Let $\mathbf{A}$ be an MV-algebra, s, t semi-states on $\mathbf{A}$. Then $t \leq s$ iff $t(x) = 1$ implies $s(x) = 1$ for all $x \in A$.

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Let $\mathbf{A}$ be an MV-algebra, t a semi-state on $\mathbf{A}$ and $S_t = \{s : \mathbf{A} \to [0,1] \mid s$ is an MV-morphism, $s \geq t\}$. Then $t = \bigwedge S_t$.

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Corollary

Any semi-state on an MV-algebra $\mathbf{A}$ is a strong semi-state.

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The only semi-state s on an MV-algebra $\mathbf{A}$ with $s(0) \neq 0$ is the constant function $s(x) = 1$ for all $x \in A$.

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The only semi-state s on the standard MV-algebra $[0,1]$ with $s(0) = 0$ is the identity function.

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The dual version

Remark

It is transparent that all the preceding notions and results including Proposition 8 can be dualized. In particular, any dual semi-state, i.e., a map $s : \mathbf{A} \to [0,1]$ *satisfying conditions (i),(ii),(iv), (v) and the dual condition (iii)'* $s(x) = 0$ *and* $s(y) = 0$ *implies* $s(x \oplus y) = 0$ *is a join of extremal states on* $\mathbf{A}$*.*

Proposition

Let $\mathbf{A}$ *be an MV-algebra,* s *a state on* $\mathbf{A}$*. Then the following conditions are equivalent:*

- (a) *s is a morphism of MV-algebras,*
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Outline

Functions between MV-algebras

This section studies the notion of an fm-function between MV-algebras (strong fm-function between MV-algebras). The main purpose of this section is to establish in some sense a canonical construction of strong fm-function between MV-algebras. This construction is an ultimate source of numerous examples.

Definition

By an **fm-function between MV-algebras** G is meant a function $G : \mathbf{A}_1 \to \mathbf{A}_2$ such that $\mathbf{A}_1 = (A_1; \oplus_1, \odot_1, \neg_1, 0_1, 1_1)$ and $\mathbf{A}_2 = (A_2; \oplus_2, \odot_2, \neg_2, 0_2, 1_2)$ are MV-algebras and

(FM1) $G(1_1) = 1_2$,

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(FM10) $n \times_2 G(x) = G(n \times_1 x)$ for all $n \in \mathbb{N}$,

we say that G is a **strong fm-function between MV-algebras**.

If $G : \mathbf{A}_1 \to \mathbf{A}_2$ and $H : \mathbf{B}_1 \to \mathbf{B}_2$ are fm-functions between MV-algebras, then a *morphism between G and H* is a pair (φ, ψ) of morphism of MV-algebras $\varphi : \mathbf{A}_1 \to \mathbf{B}_1$ and $\psi : \mathbf{A}_2 \to \mathbf{B}_2$ such that $\psi(G(x)) = H(\varphi(x))$, for any $x \in A_1$.

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Note that (FM8) yields (FM2), (FM9) yields (FM4) and (FM10) yields (FM5). Also, a composition of fm-functions (strong fm-functions) is an fm-function (a strong fm-function) again and any morphism of MV-algebras is an fm-function (a strong fm-function).

The notion of an fm-function generalizes both the notions of a semi-state and of a $\odot$-operator which is an fm-function G from $\mathbf{A}_1$ to itself such that (FM6) is satisfied.

According to both (FM4) and (FM5), $G|_{B(\mathbf{A}_1)} : B(\mathbf{A}_1) \to B(\mathbf{A}_2)$ is an fm-function (a strong fm-function) whenever G has the respective property.

Lemma

Let $G : \mathbf{A}_1 \to \mathbf{A}_2$ be an fm-function between MV-algebras, $r \in (0,1) \cap \mathbb{D}$. Then $t_r(G(x)) = G(t_r(x))$ for all $x \in A_1$.

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The construction of strong functions between MV-algebras I

By a **frame** is meant a triple (S,T,R) where S,T are non-void sets and $R \subseteq S \times T$.

Having an MV-algebra $\mathbf{M} = (M; \oplus, \odot, \neg, 0, 1)$ and a non-void set T, we can produce the direct power $\mathbf{M}^T = (M^T; \oplus, \odot, \neg, o, j)$ where the operations $\oplus$, $\odot$ and $\neg$ are defined and evaluated on $p, q \in M^T$ componentwise. Moreover, o, j are such elements of M^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathbf{M}^T$ is again an MV-algebra.

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The construction of strong functions between MV-algebras II

Theorem

Let $\mathbf{M}$ be a linearly ordered complete MV-algebra, (S,T,R) be a frame and G^* be a map from M^T into M^S defined by

$$G^*(p)(s) = \bigwedge\{p(t) \mid t \in T, sRt\},$$

for all $p \in M^T$ and $s \in S$. Then G^* is a strong fm-function between MV-algebras which has a left adjoint P^*.

In this case, for all $q \in M^S$ and $t \in T$,

$$P^*(q)(t) = \bigvee\{q(s) \mid s \in S, sRt\}$$

and $P^* : (\mathbf{M}^S)^{op} \to (\mathbf{M}^T)^{op}$ is a strong fm-function between MV-algebras.

We say that $G^* : \mathbf{M}^T \to \mathbf{M}^S$ is the *canonical strong fm-function between MV-algebras induced by the frame (S,T,R) and the MV-algebra $\mathbf{M}$.*

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Outline

Semisimple MV-algebras

Recall that

1. *semisimple* MV-algebras are just subdirect products of the simple MV-algebras,
2. any simple MV-algebra is uniquelly embeddable into the standard MV-algebra on the interval $[0,1]$ of reals,
3. an MV-algebra is semisimple if and only if the intersection of the set of its maximal (prime) filters is equal to the set $\{1\}$,
4. any complete MV-algebra is semisimple.

A semisimple MV-algebra $\mathbf{A}$ is embedded into $[0,1]^T$ where T is the set of all ultrafilters of $\mathbf{A}$ (morphisms from $\mathbf{A}$ into the standard MV-algebra) and $\pi_F(x) = x(F) = x/F \in [0,1]$ for any $x \in \mathbf{S} \subseteq [0,1]^T$ and any $F \in T$; here $\pi_F : [0,1]^T \to [0,1]$ is the respective projection onto $[0,1]$.

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Let $G : \mathbf{A}_1 \to \mathbf{A}_2$ be an fm-function between semisimple MV-algebras, T (S) a set of all MV-morphism from $\mathbf{A}_1$ ($\mathbf{A}_2$) into the standard MV-algebra $[0,1]$. Further, let (S, T, ρ_G) be a frame such that the relation $\rho_G \subseteq S \times T$ is defined by

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For any MV-algebra $\mathbf{A}_1$, any semisimple MV-algebra $\mathbf{A}_2$ with a set S of all MV-morphism from $\mathbf{A}_2$ to $[0,1]$ and any map $G : A_1 \to A_2$ the following conditions are equivalent:

(i) *G is an fm-function between MV-algebras.*

(ii) *G is a strong fm-function between MV-algebras.*

Proposition

There is an MV-algebra $\mathbf{A}$ with an fm-function G on $\mathbf{A}$ such that G is not a strong fm-function.

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Tense operators on MV-algebras

Definition (Botur and Paseka, Diaconescu and Georgescu)

Let $\mathscr{M}$ be an MV-algebra with (strong) fm-functions G and H on $\mathscr{M}$. The structure $(\mathscr{M};G,H)$ is called a *(strong) tense MV-algebra* if the following condition is fulfilled:

(GH) $\quad x \leq G(\neg H(\neg x)), \quad x \leq H(\neg G(\neg x)),$ for all $x \in M$.

Corollary

For any semisimple MV-algebra $\mathscr{M}$ and any maps $G,H : M \to M$ the following conditions are equivalent:

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(ii) $(\mathscr{M};G,H)$ *is a strong tense MV-algebra.*

Tense operators on MV-algebras

Definition (Botur and Paseka, Diaconescu and Georgescu)

Let $\mathscr{M}$ be an MV-algebra with (strong) fm-functions G and H on $\mathscr{M}$. The structure $(\mathscr{M}; G, H)$ is called a *(strong) tense MV-algebra* if the following condition is fulfilled:

(GH) $$x \leq G(\neg H(\neg x)), \quad x \leq H(\neg G(\neg x)), \text{ for all } x \in M.$$

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Tense operators on MV-algebras– motivation

G	"It will always be the case that ..."
$P = \neg \circ H \circ \neg$	"It has at some time been the case that ..."
H	"It has always been the case that ..."
$F = \neg \circ G \circ \neg$	"It will at some time be the case that ..."

P and F are known as the *weak tense operators*, while H and G are known as the *strong tense operators*.

Moreover, P is a left adjoint to G and F is a left adjoint to H.

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Diaconescu and Georgescu formulated the following open problem:

Characterize those (strong) tense MV-algebras $(\mathscr{M};G,H)$ *such that* $i_{\mathscr{M}} : (\mathscr{M};G,H) \to ([0,1]^T;G^*,H^*)$ *is a morphism of tense MV-algebra.*

Theorem (Representation theorem for tense MV-algebras)

For any semisimple tense MV-algebra $(\mathscr{M};G,H)$*,* $(\mathscr{M};G,H)$ *is embeddable via the morphism* $i_{\mathscr{M}}$ *of tense MV-algebras into the canonical tense MV-algebra* $\mathscr{L}_{G,H} = ([0,1]^T;G^*,H^*)$ *with strong operators* G^*,H^* *induced by the canonical frames* (T,R_G)*,* (T,R_H) *and the standard MV-algebra* $[0,1]$*.*

Further, for all $x \in M$ *and for all* $s \in T$*,* $s(G(x)) = G^*((t(x))_{t\in T})(s)$ *and* $s(H(x)) = H^*((t(x))_{t\in T})(s)$*.*

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$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{\ G\ } & \mathbf{M} \\ \downarrow{\scriptstyle i^T_{\mathbf{M}}} & & \downarrow{\scriptstyle i^T_{\mathbf{M}}} \\ [0,1]^T & \xrightarrow[G^*]{} & [0,1]^T \end{array} \quad .$$

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Thank you for your attention.