# Factoraggregation based on fuzzy equivalence relation

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# Outline

- Factoraggregation
- Generalized factoraggregation
- Numerical examples
- Lower generalized factoraggregation
- Case of crisp equivalence relation

# Aggregation operator

**Definition**

A mapping $A : \bigcup_n [0,1]^n \to [0,1]$ is called an aggregation operator, if it satisfies:

$(A1)$ $A(0, \ldots, 0) = 0$;

$(A2)$ $A(1, \ldots, 1) = 1$;

$(A3)$ $\forall x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in [0,1]$:
if $x_1 \leq y_1, \ldots, x_n \leq y_n$, then $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$.

(A1) un (A2) – boundary conditions;

(A3) – monotonicity condition.

# General aggregation operator

## Definition

*(A. Takaci, 2003)* A mapping $\tilde{A}\colon \bigcup_n([0,1]^D)^n \to [0,1]^D$ is called a general aggregation operator if the following conditions hold:

($\tilde{A}1$) $\tilde{A}(\tilde{0},\dots,\tilde{0}) = \tilde{0}$;

($\tilde{A}2$) $\tilde{A}(\tilde{1},\dots,\tilde{1}) = \tilde{1}$;

($\tilde{A}3$) $\forall \mu_1,\mu_2,\dots,\mu_n,\eta_1,\eta_2,\dots,\eta_n \in [0,1]^D$ :
if $\mu_1 \preceq \eta_1,\dots,\mu_n \preceq \eta_n$, then $\tilde{A}(\mu_1,\dots,\mu_n) \preceq \tilde{A}(\eta_1,\dots,\eta_n)$.

Here $\mu_1,\mu_2,...,\mu_n,\eta_1,\eta_2,...,\eta_n \in [0,1]^D$ are fuzzy sets, $\preceq$ is an order on $[0,1]^D$, while $\tilde{0}$, $\tilde{1}$ are indicators of $\varnothing$ and $D$ respectively, i.e.

$$\tilde{0}(x) = 0 \text{ and } \tilde{1}(x) = 1 \text{ for all } x \in D.$$

# Factoraggregation

Let $A\colon [0,1]^n \to [0,1]$ be an ordinary aggregation operator and $\rho$ be an equivalence relation defined on a set $D$. An operator

$$\tilde{A}_\rho\colon \bigcup_n ([0,1]^D)^n \to [0,1]^D$$

such as

$$\tilde{A}_\rho(\mu_1, \mu_2, \ldots, \mu_n)(x) = \sup_{u \in D : (u,x) \in \rho} A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u)),$$

where $x \in D$ and $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$, is called a factoraggregation operator corresponding to $\rho$.

# Generalized factoraggregation

Let $T$ be a t-norm, $E$ be a $T$-fuzzy equivalence relation defined on $D$ and $A$ be an ordinary aggregation operator. An operator

$$\tilde{A}_{E,T} \colon \bigcup_{n} ([0,1]^D)^n \to [0,1]^D$$

such as

$$\tilde{A}_{E,T}(\mu_1, \mu_2, \ldots, \mu_n)(x) = \sup_{u \in D} T(E(x,u), A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u))),$$

where $x \in D$ and $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$, is called a generalized $T$-fuzzy factoraggregation operator corresponding to $E$.

# $T$-fuzzy equivalence relation

**Definition**

Let $T$ be a t-norm and $E$ be a fuzzy relation on a set $D$, i.e. $E$ is a fuzzy subset of $D \times D$. A fuzzy relation $E$ is a $T$-fuzzy equivalence relation if and only if for all $x, y, z \in D$ it holds

$(E1)$ $E(x, x) = 1$ (reflexivity);

$(E2)$ $E(x, y) = E(y, x)$ (symmetry);

$(E3)$ $T(E(x, y), E(y, z)) \leq E(x, z)$ ($T$-transitivity).

# Generalized factoraggregation: numerical examples

Let us consider the discrete universe

$$D = \{x_1, x_2, x_3, x_4, x_5\}$$

and the following $T_L$-fuzzy ($T_L$ is the Lukasiewicz t-norm) equivalence relation $E$, given in the matrix form:

$$E = \begin{pmatrix} 1 & 0.9 & 0.7 & 0.4 & 0.2 \\ 0.9 & 1 & 0.7 & 0.4 & 0.2 \\ 0.7 & 0.7 & 1 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.4 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 1 \end{pmatrix}.$$

This equivalence relation is also $T_M$-transitive and $T_P$-transitive, i.e. transitive with respect to the minimum t-norm $T_M$ and the product $t$-norm $T_P$ respectively.

# Generalized factoraggregation: numerical examples

Let us take the following fuzzy subsets of $D$:

$$\mu_1 = \begin{pmatrix} 0.9 \\ 0.5 \\ 0.6 \\ 0.8 \\ 0.3 \end{pmatrix}, \ \mu_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.2 \\ 0.6 \\ 0.9 \end{pmatrix}, \ \mu_3 = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.1 \\ 0.8 \\ 0.6 \end{pmatrix}, \ \mu_4 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.2 \\ 0.8 \\ 0.5 \end{pmatrix}.$$

# Generalized factoraggregation: numerical examples

We consider the minimum aggregation operator $A = MIN$ and obtain the following generalized $T$-fuzzy factoraggregation:

$$\tilde{A}_{E,T}(\mu_1, \mu_2, \mu_3, \mu_4)(x) =$$

$$= \max_{u \in D} T(E(x, u), \min(\mu_1(u), \mu_2(u), \mu_3(u), \mu_4(u))).$$

Taking $T = T_L$, $T = T_M$ and $T = T_P$ we obtain as results the fuzzy subsets $\mu_{T_L}$, $\mu_{T_M}$ and $\mu_{T_P}$ respectively:

$$\mu_{T_L} = \begin{pmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.6 \\ 0.3 \end{pmatrix}, \ \mu_{T_M} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \end{pmatrix}, \ \mu_{T_P} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.6 \\ 0.3 \end{pmatrix}.$$

# Generalized factoraggregation: numerical examples

Taking as an ordinary aggregation operator the arithmetic mean aggregation operator $A = AVG$, we obtain the following generalized $T$-fuzzy factoraggregations respectively:

$$\tilde{A}_{E,T}(\mu_1, \mu_2, \mu_3, \mu_4)(x) =$$

$$= \max_{u \in D} T(E(x, u), AVG(\mu_1(u), \mu_2(u), \mu_3(u), \mu_4(u))),$$

Taking $T = T_L$, $T = T_M$ and $T = T_P$ we obtain as results the following fuzzy subsets:

$$\mu_{T_L} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.3 \\ 0.8 \\ 0.6 \end{pmatrix}, \ \mu_{T_M} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.8 \\ 0.6 \end{pmatrix}, \ \mu_{T_P} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.3 \\ 0.8 \\ 0.6 \end{pmatrix}.$$

# Generalized factoraggregation and extensional fuzzy sets

### Definition

Let $T$ be a t-norm and $E$ be a $T$-fuzzy equivalence relation on a set $D$. A fuzzy subset $\mu \in [0,1]^D$ is called extensional with respect to $E$ if and only if:

$$T(E(x,y),\mu(y)) \leq \mu(x) \text{ for all } x,y \in D.$$

### Proposition

Let $T$ be a left-continuous t-norm, $E$ be a $T$-fuzzy equivalence relation on a set $D$ and $\tilde{A}_{E,T}$ be a generalized $T$-fuzzy factoraggregation. Then fuzzy set $\tilde{A}_{E,T}(\mu_1,\mu_2,\ldots,\mu_n)$ is extensional with respect to $E$ for each $n \in \mathbb{N}$ and for all fuzzy sets $\mu_1, ..., \mu_n \in [0,1]^D$.

# Approximation of a fuzzy set by extensional fuzzy sets

We recall two approximation operators $\phi_E$ and $\psi_E$ considered in [see e.g. Mattioli, Recasens, AGOP 2013]. Fuzzy sets $\phi_E(\mu)$ and $\psi_E(\mu)$ were introduced to provide upper and lower approximation of a fuzzy set $\mu$ by extensional fuzzy sets with respect to $T$-fuzzy equivalence relation $E$

**Definition**

Let $T$ be a left-continuous t-norm, $\vec{T}$ be its residuum and $E$ be a $T$-fuzzy equivalence relation on a set $D$. The maps $\phi_E\colon [0,1]^D \to [0,1]^D$ and $\psi_E\colon [0,1]^D \to [0,1]^D$ are defined by:

$$\phi_E(\mu)(x) = \sup_{y \in D} T(E(x,y), \mu(y)),$$

$$\psi_E(\mu)(x) = \inf_{y \in D} \vec{T}(E(x,y)|\mu(y))$$

for all $x \in D$ and for all $\mu \in [0,1]^D$.

# Lower generalized $T$-fuzzy factoraggregation

Let $T$ be a left-continuous t-norm, $\overrightarrow{T}$ be the residuum of $T$, $E$ be a $T$-fuzzy equivalence relation defined on $D$ and $A$ be an ordinary aggregation operator. An operator

$$\tilde{A}_{E,\overrightarrow{T}} : \bigcup_n ([0,1]^D)^n \to [0,1]^D$$

such as

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1, \mu_2, \ldots, \mu_n)(x) = \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u))),$$

where $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$ and $x \in D$, is called a lower generalized $T$-fuzzy factoraggregation operator corresponding to $E$.

# Lower generalized $T$-fuzzy factoraggregation

It is clear, that for all $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$ and for all $x \in D$ it holds

$$\tilde{A}_{E,\vec{T}}(\mu_1, \ldots, \mu_n)(x) \leq A(\mu_1(x), \ldots, \mu_n(x)) \leq \tilde{A}_{E,T}(\mu_1, \ldots, \mu_n)(x).$$

**Proposition**

Let $T$ be a left-continuous t-norm, $E$ be a $T$-fuzzy equivalence relation on a set $D$ and $\tilde{A}_{E,\vec{T}}$ be a lower generalized $T$-fuzzy factoraggregation. Then fuzzy set $\tilde{A}_{E,\vec{T}}(\mu_1, \mu_2, \ldots, \mu_n)$ is extensional with respect to $E$ for each $n \in \mathbb{N}$ and for all fuzzy sets $\mu_1, ..., \mu_n \in [0,1]^D$.

# Lower generalized factoraggregation: numerical examples

Similarly to the case of upper generalized factoraggregation, we will calculate several numerical results for the following lower generalized $T$-fuzzy factoraggregation:

$$\widetilde{A}_{E,\overrightarrow{T}}(\mu_1,\mu_2,\mu_3,\mu_4)(x) =$$

$$= \min_{u \in D} \overrightarrow{T}(E(x,u)|AVG(\mu_1(u),\mu_2(u),\mu_3(u),\mu_4(u))) :$$

As a result we obtain the fuzzy subsets $\mu_{\overrightarrow{T}_L}$, $\mu_{\overrightarrow{T}_M}$ and $\mu_{\overrightarrow{T}_P}$:

$$\mu_{\overrightarrow{T}_L} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.3 \\ 0.8 \\ 0.6 \end{pmatrix}, \ \mu_{\overrightarrow{T}_M} = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.6 \end{pmatrix}, \ \mu_{\overrightarrow{T}_P} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.3 \\ 0.7 \\ 0.6 \end{pmatrix}.$$

# Generalized factoraggregations: the case of a crisp relation

Let us note that in the case of crisp equivalence relations, i.e. when $E = E_\rho$ for an equivalence relation $\rho$, where

$$E_\rho(x,y) = \begin{cases} 1, & (x,y) \in \rho, \\ 0, & (x,y) \notin \rho, \end{cases}$$

we obtain

$$\tilde{A}_{E_\rho,T} = \tilde{A}_\rho.$$

If we apply the crisp equivalence relation $E_\rho$ to $\tilde{A}_{E_\rho,\vec{T}}$, for any left-continuous t-norm $T$ we obtain the following formula:

$$\tilde{A}_{E_\rho,\vec{T}}(\mu_1,\ldots,\mu_n)(x) = \inf_{u \in D:(u,x)\in\rho} A(\mu_1(u),\ldots,\mu_n(u))$$

for all $\mu_1,\mu_2,\ldots,\mu_n \in [0,1]^D$ and $x \in D$.

# Generalized factoraggregations: the case of a crisp relation

Numerical evaluation of the value $\tilde{A}_{E_\rho,T}(\mu_1, ..., \mu_n)(x)$ can be reduced to the problem

$$\alpha \longrightarrow \text{min}$$

$$\begin{cases} A(\mu_1(u), \ldots, \mu_n(u)) \leq \alpha, \\ (u, x) \in \rho, \quad u \in D. \end{cases}$$

By analogy with the previous case, numerical evaluation of the value $\widetilde{A}_{E_\rho, \vec{T}}(\mu_1, ..., \mu_n)(x)$ can be reduced to the problem

$$\alpha \longrightarrow \text{max}$$

$$\begin{cases} A(\mu_1(u), \ldots, \mu_n(u)) \geq \alpha, \\ (u, x) \in \rho, \quad u \in D. \end{cases}$$

Thank you for your attention!