Laws of contraposition and law of importation for probabilistic implications and probabilistic S-implications

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Content

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- Basic definitions.

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- Introduction of Probabilistic Implications and Probabilistic S-implications.

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- Laws of Contraposition.

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- Law of Importation.

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- Introduction of Probabilistic Implications and Probabilistic S-implications.
- Laws of Contraposition.
- Law of Importation.
- What does still remain to do?

Basic definitions

Definition 1

A function $I : [0,1]^2 \to [0,1]$ *is called a* ***fuzzy implication*** *if it satisfies the following conditions*

(I1) *if* $I(\_, y)$ *is decreasing;*

(I2) *if* $I(x, \_)$ *is increasing;*

(I3) $I(0,0) = 1$; *(I4)* $I(1,1) = 1$; *(I5)* $I(1,0) = 0$.

Basic definitions

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(I3) $I(0,0) = 1$; *(I4)* $I(1,1) = 1$; *(I5)* $I(1,0) = 0$.

Definition 2

*An associative, commutative and non-decreasing operation $T : [0,1]^2 \to [0,1]$ is called a **t-norm** if it nas the neutral element 1.*

Definition 3

*A non-increasing function $N : [0,1]^2 \to [0,1]$ is called a **fuzzy negation** if $N(0) = 1, N(1) = 0$.*
*A fuzzy negation is called **strong** if it is an involution, i.e. $N(N(x)) = x$ for all $x \in [0,1]$.*

Uncertainty

Cause of uncertainty of knowledge:

- random effects
- imprecision or fuzziness

Uncertainty

Cause of uncertainty of knowledge:

- random effects → probability theory
- imprecision or fuzziness → fuzzy theory

Derivation of Probabilistic Implications

$$\boxed{A \rightarrow B \quad \equiv \quad P(B|A)}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \quad (P(A) > 0)$$

Derivation of Probabilistic Implications

$$\boxed{A \to B \quad \equiv \quad P(B|A)}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \quad (P(A) > 0)$$

$$\boxed{A \to B \quad \equiv \quad \neg A \vee B}$$

$$\begin{aligned} P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) \\ &= P(A') + P(A \cap B) \\ &= P(A \cap B) - P(A) + 1 \end{aligned}$$

Copulas

Definition 4

*A **copula** (specifically a 2-copula) is a function* $C : [0,1]^2 \to [0,1]$ *which satisfies the following conditions:*

(a) $C(u,0) = C(0,v) = 0$ *for every* $u, v \in [0,1]$

(b) $C(u,1) = u$ *for every* $u \in [0,1]$

(c) $C(1,v) = v$ *for every* $v \in [0,1]$

(d) *for every* $u_1, u_2, v_1, v_2 \in [0,1]$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Copulas

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(c) $C(1,v) = v$ *for every* $v \in [0,1]$
(d) *for every* $u_1, u_2, v_1, v_2 \in [0,1]$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Theorem 1

Let X *and* Y *be random variables with joint distribution function* H *and marginal distribution functions* F *and* G*, respectively. Then there exists a copula* C *such that*

$$H(x,y) = C(F(x), G(y)) \tag{1}$$

for all $x, y \in \mathbb{R}$.
Conversely, if C *is a copula and* F *and* G *are distribution functions, then the function* H *defined by (1) is a joint distribution function with margins* F *and* G.

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$$H(x,y) = C(F(x),G(y))$$

$$P(Y \leq y | X \leq x) = \frac{P(X \leq x, Y \leq y)}{P(X \leq x)} = \frac{H(x,y)}{F(x)} = \frac{C(F(x),G(y))}{F(x)} = \frac{C(u,v)}{u},$$

where $u = F(x)$ and $v = G(y)$ ($u, v \in [0,1]$).

Derivation of Probabilistic Implications

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where $u = F(x)$ and $v = G(y)$ $(u, v \in [0,1])$.

Grzegorzewski, EUSFLAT 2011:

Definition 5

A function $I_C : [0,1]^2 \to [0,1]$ *given by*

$$I_C(u,v) = \begin{cases} 1 & \text{if } u = 0 \\ \frac{C(u,v)}{u} & \text{if } u > 0, \end{cases} \tag{2}$$

where C is a copula, is called a ***probabilistic implication*** *(based on copula C).*

Derivation of Probabilistic S-Implications

$P(A' \cup B) = P(A \cap B) - P(A) + 1$

$A = \{\omega : X(\omega) \leq x\}$
$B = \{\omega : Y(\omega) \leq y\}$

$H(x,y) = C(F(x),G(y))$

Derivation of Probabilistic S-Implications

$P(A' \cup B) =$
$P(A \cap B) - P(A) + 1$

$A = \{\omega : X(\omega) \leq x\}$
$B = \{\omega : Y(\omega) \leq y\}$

$H(x,y) = C(F(x),G(y))$

$$
\begin{aligned}
P(X > x \text{ or } Y \leq y) &= P(X \leq x, Y \leq y) - P(X \leq x) + 1 \\
&= H(x,y) - F(x) + 1 \\
&= C(F(x), G(y)) - F(x) + 1 = C(u,v) - u + 1
\end{aligned}
$$

where $u = F(x)$ and $v = G(y)$ $\quad (u, v \in [0,1])$

Derivation of Probabilistic S-Implications

P(A' ∪ B) =
P(A ∩ B) - P(A) + 1

A = {ω : X(ω) ≤ x}
B = {ω : Y(ω) ≤ y}

H(x,y) = C(F(x),G(y))

$$\begin{aligned} P(X > x \text{ or } Y \leq y) &= P(X \leq x, Y \leq y) - P(X \leq x) + 1 \\ &= H(x,y) - F(x) + 1 \\ &= C(F(x), G(y)) - F(x) + 1 = C(u,v) - u + 1 \end{aligned}$$

where $u = F(x)$ and $v = G(y)$ $\quad (u, v \in [0,1])$

Grzegorzewski, EUSFLAT 2011:

Definition 6

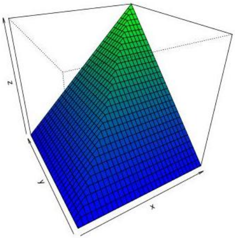
A function $I_C : [0,1]^2 \to [0,1]$ given by

$$\widetilde{I}_C(u,v) = C(u,v) - u + 1 \tag{3}$$

where C is a copula, is called a ***probabilistic S-implication*** *(based on copula C).*

Examples - C = min

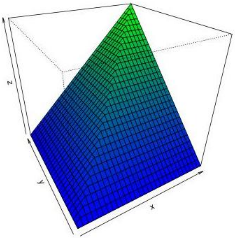
$C(u, v) = M(u, v) = \min(u, v)$

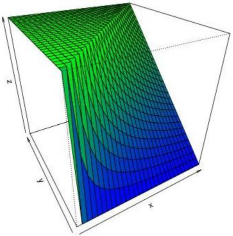


Examples - C = min

$C(u,v) = M(u,v) = \min(u,v)$

$I_M(u,v) = I_{GG}(u,v) = \begin{cases} 1 & \text{if } u \le v \\ \frac{v}{u} & \text{if } u > v \end{cases}$





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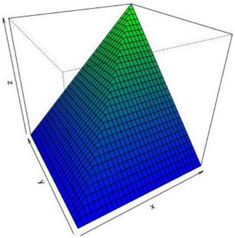
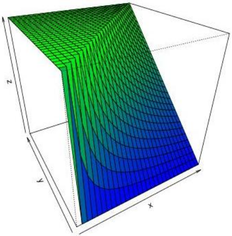
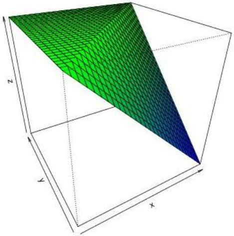
$\tilde{I}_M(u,v) = I_{LK}(u,v) = \min(1, 1-u+v)$





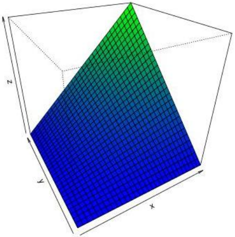


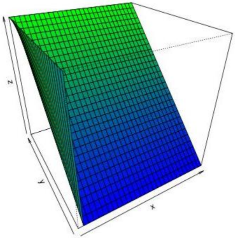
Examples - $C = \Pi$

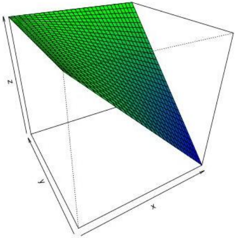
$$C(u,v) = \Pi(u,v) = uv$$

$$I_\Pi(u,v) = \begin{cases} 1 & \text{if } u = 0 \\ v & \text{if } u > 0 \end{cases}$$

$$\tilde{I}_\Pi(u,v) = I_{RC}(u,v) = uv - u + 1$$





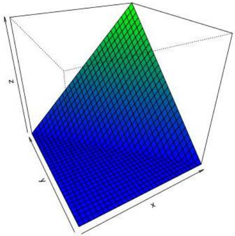


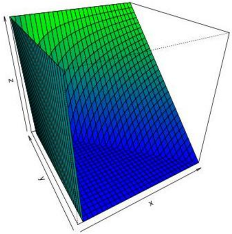
Examples - C = W

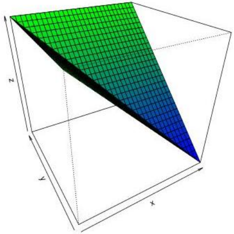
$C(u,v) = W(u,v) = \max(u+v-1,0)$

$I_W(u,v) = \begin{cases} 1 & \text{if } u = 0 \\ \frac{\max(u+v-1,0)}{u} & \text{if } u > 0 \end{cases}$

$\tilde{I}_W(u,v) = I_{KD}(u,v) = \max(1-u,v)$







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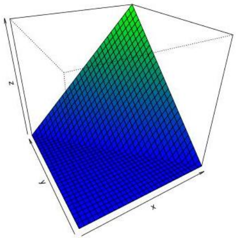
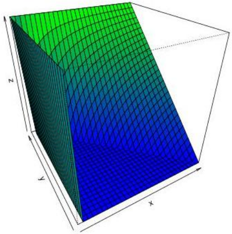
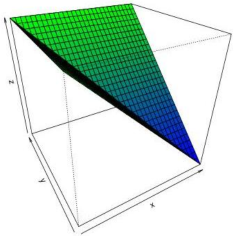




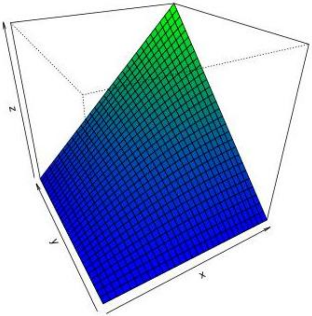


I is a probabilistic S-implication $\to$ I is a fuzzy implication

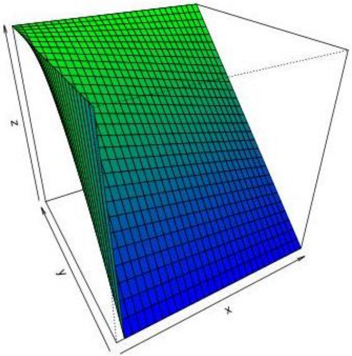
I is a probabilistic implication + (I1) $\to$ I is a fuzzy implication

Examples - $C \in FGM(\Theta)$

$C_{FGM(\Theta)}(u, v) =$
$uv + \Theta uv(1-u)(1-v)$

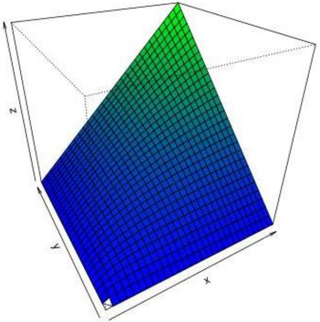


$I_{C(FGM(\Theta))}(u, v) =$
$\begin{cases} 1 & \text{if } u = 0 \\ v + \Theta v(1-u)(1-v) & \text{if } u > 0 \end{cases}$

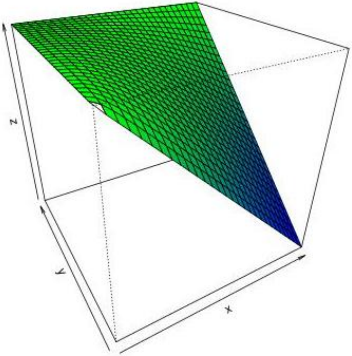


Examples - $C \in AMH(\Theta)$

$$C_{AMH(\Theta)}(u, v) = \frac{uv}{1-\Theta(1-u)(1-v)}$$



$$\tilde{I}_{AMH(\Theta)}(u, v) = \frac{uv}{1-\Theta(1-u)(1-v)} - u + 1$$



Main properties of Probabilistic Implications

	Probabilistic Implications	Probabilistic S-Implications
(NP)	✓	✓
(IP)	✓ ONLY FOR I_{GG} (the Goguen implication - based on copula M=min)	✓ ONLY FOR I_{LK} (the Łukasiewicz implication - based on copula M=min)
(OP)	✓ ONLY FOR I_{GG}	✓ ONLY FOR I_{LK}
(EP)	✓ FOR SOME (e.g. I_{GG}, I_{Π}) ✗ FOR SOME (e.g. $I_{FGM(\theta)}, \theta \neq 0$)	✓ FOR SOME (e.g. I_{LK}, I_{RC}) ✗ FOR SOME (e.g. some $I_{AMH(\theta)}$)

NEW
RESULTS

Laws of Contraposition - definition

$$\boxed{p \to q \equiv \neg q \to \neg p}$$

Laws of Contraposition - definition

$\boxed{p \rightarrow q \equiv \neg q \rightarrow \neg p}$ $\longrightarrow$ $\neg(\neg p) \equiv p$

$\boxed{\neg p \rightarrow q \equiv \neg q \rightarrow p}$

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Laws of Contraposition - definition

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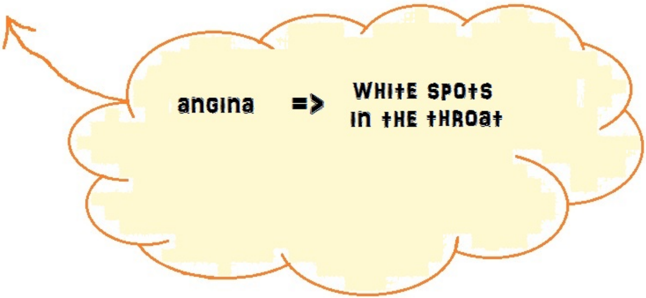

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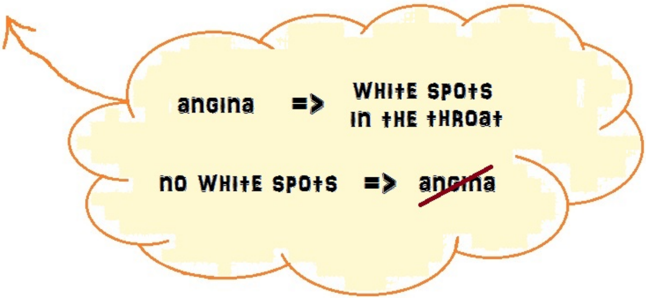

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Definition 7

I - fuzzy implication, N - fuzzy negation. I satisfies the

1. ***Law of Contraposition** with respect to N, if*

$$I(x,y) = I(N(y), N(x)) \quad \forall_{x,y \in [0,1]} \quad \textbf{(CP),(CP(N))}.$$

Laws of Contraposition - definition

$\boxed{p \to q \equiv \neg q \to \neg p} \quad \longrightarrow \quad \neg(\neg p) \equiv p$

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1. ***Law of Contraposition** with respect to N, if*

$$I(x, y) = I(N(y), N(x)) \quad \forall_{x,y \in [0,1]} \quad \textbf{(CP),(CP(N))}.$$

2. ***Law of Left Contraposition** with respect to N, if*

$$I(N(x), y) = I(N(y), x) \quad \forall_{x,y \in [0,1]} \quad \textbf{(L-CP),(L-CP(N))}.$$

Laws of Contraposition - definition

$\boxed{p \to q \equiv \neg q \to \neg p}$ $\longrightarrow$ $\neg(\neg p) \equiv p$ $\boxed{\neg p \to q \equiv \neg q \to p}$ $\boxed{p \to \neg q \equiv q \to \neg p}$

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1. ***Law of Contraposition** with respect to N, if*

$$I(x,y) = I(N(y), N(x)) \quad \forall_{x,y \in [0,1]} \quad \textbf{(CP),(CP(N))}.$$

2. ***Law of Left Contraposition** with respect to N, if*

$$I(N(x),y) = I(N(y), x) \quad \forall_{x,y \in [0,1]} \quad \textbf{(L-CP),(L-CP(N))}.$$

3. ***Law of Right Contraposition** with respect to N, if*

$$I(x,N(y)) = I(y, N(x)) \quad \forall_{x,y \in [0,1]} \quad \textbf{(R-CP),(R-CP(N))}.$$

Laws of Contraposition for Probabilistic Implications

Lemma 1

No probabilistic implication satisfies the Laws of Contraposition (CP) and (L-CP) with respect to any negation N.

Laws of Contraposition for Probabilistic Implications

Lemma 1

No probabilistic implication satisfies the Laws of Contraposition (CP) and (L-CP) with respect to any negation N.

Lemma 2

Every probabilistic implication satisfies $(R - CP)$, but only with respect to the least negation $N_{D1}(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$.

Laws of Contraposition for Probabilistic S-Implications

Laws of Contraposition for Probabilistic S-Implications

Lemma 3

Let $\tilde{I}_C$ be any probabilistic S-implication. If $\tilde{I}_C$ satisfies the (CP) with respect to a fuzzy negation N, then N is the strong negation $N_C(x) = 1 - x$.

Laws of Contraposition for Probabilistic S-Implications

Lemma 3

Let $\tilde{I}_C$ be any probabilistic S-implication. If $\tilde{I}_C$ satisfies the (CP) with respect to a fuzzy negation N, then N is the strong negation $N_C(x) = 1 - x$.

Corollary 1

Probabilistic S-implication $\tilde{I}_C$ based on a copula C satisfies (CP) (with respect to N_C) if and only if C satisfies the following equation

$$C(x, y) - x + 1 = C(1 - y, 1 - x) + y, \tag{4}$$

for all $x, y \in (0, 1)$.

Laws of Contraposition for Probabilistic S-Implications

Lemma 3

Let $\tilde{I}_C$ be any probabilistic S-implication. If $\tilde{I}_C$ satisfies the (CP) with respect to a fuzzy negation N, then N is the strong negation $N_C(x) = 1 - x$.

Corollary 1

Probabilistic S-implication $\tilde{I}_C$ based on a copula C satisfies (CP) (with respect to N_C) if and only if C satisfies the following equation

$$C(x, y) - x + 1 = C(1 - y, 1 - x) + y, \tag{4}$$

for all $x, y \in (0, 1)$.

Example 1

$C = \Pi$, $C = M$, $C = W$, $C \in FGM(\Theta)$ *OK*
$C \in AMH(\Theta)$ *X*

Law of Importation - definition

$$\boxed{(p \wedge q) \to r \equiv p \to (q \to r)}$$

Law of Importation - definition

$$\boxed{(p \wedge q) \to r \equiv p \to (q \to r)}$$

Definition 8

Let I be a fuzzy implication and T be a t-norm. I is said to satisfy the ***Law of Importation*** *with t-norm T, if*

$$I(T(x,y),z) = I(x, I(y,z)) \quad \forall_{x,y,z \in [0,1]} \qquad \textbf{\textit{(LI)}}.$$

Law of Importation for Probabilistic Implications

Law of Importation for Probabilistic Implications

Lemma 4

If a probabilistic implication I_C and a t-norm T satisfy the law of importation (LI), then T is positive, i.e., $\neg\exists_{x,y\neq 0} T(x,y) = 0$.

Law of Importation for Probabilistic Implications

Lemma 4

If a probabilistic implication I_C and a t-norm T satisfy the law of importation (LI), then T is positive, i.e., $\neg\exists_{x,y\neq 0} T(x,y) = 0$.

Example 2

- I_Π - *with any positive t-norm* T.
- I_M - *only with* T_P.
- I_W - *only with* T_P.
- *FGM*(Θ) - *with any t-norm* T.
- *AMH*(Θ) - *only for* $\Theta = 1$ *and only with* T_P.

Law of Importation for Probabilistic S-Implications

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Lemma 5

If a probabilistic S-implication $\tilde{I}_C$ satisfies (LI) with any t-norm T, then T must be of the form $T(x, y) = x - C(x, 1 - y)$.

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C_Π	$T_P(x, y) = xy$	*OK*
C_M	$T_{LK}(x, y) = \max(x + y - 1, 0)$	*OK*
C_W	$T_M(x, y) = \min(x, y)$	*OK*
$C \in FGM(\Theta)$	$T(x, y) = xy - \Theta xy(1-x)(1-y)$	X
$C \in AMH(\Theta)$	$T(x, y) = xy \frac{1-\Theta(1-x)}{1-\Theta(1-x)y}$	X

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- Other properties of Probabilistic Implications and Probabilistic S-Implications may be checked still.

THANK YOU !