Convergence theorems for (semi)copula - based universal integrals

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A $[0,1]$-*valued universal integral* is a functional

$$\mathbf{I} : \bigcup_{(X,\mathscr{A})\in\mathscr{S}} \left(\mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}\right) \to [0,1]$$

satisfying the following conditions:

I1) for all $(X,\mathscr{A}) \in \mathscr{S}$, $m_1, m_2 \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $f_1, f_2 \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ with $m_1 \le m_2$, $f_1 \le f_2$ we have $\mathbf{I}(m_1, f_1) \le \mathbf{I}(m_2, f_2)$;

I2) for all $(X,\mathscr{A}) \in \mathscr{S}$, $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $A \in \mathscr{A}$ we have $m(A) = \mathbf{I}(m, \mathbf{1}_A)$;

I3) for all $(X,\mathscr{A}) \in \mathscr{S}$, $m \in \mathscr{M}^1_{(X,\mathscr{A})}$ and $c \in [0,1]$ we have $\mathbf{I}(m, c \cdot \mathbf{1}_X) = c$;

I4) $\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2)$ for all integral equivalent pairs.

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- for $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ we may introduce a single function $h_{m,f} : [0,1] \to [0,1]$ as follows

$$h_{m,f}(t) := m(\{x \in X;\ f(x) \geq t\})$$

For $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ the smallest $[0,1]$-valued universal integral having S as the underlying semicopula is given by

$$\mathbf{I}_S(m, f) := \sup_{t \in [0,1]} S(t, h_{m,f}(t)).$$

$\mathbf{I}_M$ ... the Sugeno integral
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Theorem (monotone convergence I)

Let $S \in \mathfrak{S}$ be left-continuous and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:

(i) m is continuous from below;

(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \nearrow f$, it holds $\lim_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.

Example: Consider $X =]0,1]$, $\mathscr{A} = \mathscr{B}([0,1])$ and

$$m(A) = \begin{cases} 0, & A = \emptyset \\ 1, & \text{else}, \end{cases}$$

$$f_n(x) = \begin{cases} 0, & x \in]\frac{1}{n}, 1] \\ 1, & \text{else} \end{cases}$$

for $n \in \mathbb{N}$ and $f(x) = 0$ on X. For every $n \in \mathbb{N}$ and $t \in [0,1]$ we have $h_{m,f_n}(t) = m(]0, \frac{1}{n}]) = 1$. However,

$$\mathbf{I}_S(m, f_n) = \sup_{t\in[0,1]} S(t, h_{m,f_n}(t)) = \sup_{t\in[0,1]} S(t, 1) = 1 \quad \text{and} \quad \mathbf{I}_S(m, f) = 0.$$

Theorem (monotone convergence II)

Let $S \in \mathfrak{S}$ be right-continuous and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:

(i) m is continuous from above;

(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \searrow f$, it holds $\lim\limits_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.

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Theorem (everywhere convergence)

Let $S \in \mathfrak{S}$ be continuous and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:

(i) m is continuous;

(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \to f$, it holds $\lim\limits_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.

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$$f_n \xrightarrow{m-\text{a.e.}} f \text{ iff } (\exists A \in \mathscr{A},\ m(A) = 0)\ f_n \to f \ on\ X \setminus A$$

Theorem (almost everywhere convergence)

Let $S \in \mathfrak{S}$ be continuous and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:

(i) m is null-additive and continuous;

(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \xrightarrow{m-\text{a.e.}} f$, it holds $\lim_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.

$$f_n \xrightarrow{m-\text{a.u.}} f \ \text{ iff } \ (\forall \varepsilon \in]0,1]) \ (\exists A_\varepsilon \in \mathscr{A}, \ m(A_\varepsilon) < \varepsilon) \ f_n \xrightarrow{\text{u.}} f \text{ on } X \setminus A_\varepsilon$$

- $m : \mathscr{A} \to [0,1]$ is said to be *autocontinuous from above*, iff $\lim_{n\to\infty} m(A \cup B_n) = m(A)$ for all $A \in \mathscr{A}$ and $(B_n)_1^\infty \in \mathscr{A}$ with $\lim_{n\to\infty} m(B_n) = 0$
- $m : \mathscr{A} \to [0,1]$ is said to be *autocontinuous from below*, iff $\lim_{n\to\infty} m(A \setminus B_n) = m(A)$ for all A, $(B_n)_1^\infty \in \mathscr{A}$ with $\lim_{n\to\infty} m(B_n) = 0$.

Theorem (almost uniform convergence)

Let $S \in \mathfrak{S}$ be continuous and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:

(i) m is monotone autocontinuous;

(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \xrightarrow{m-\text{a.u.}} f$, it holds $\lim_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.

$$f_n \xrightarrow{m} f \quad \text{iff} \ (\forall t \in]0,1]) \ \lim_{n\to\infty} h_{m,|f_n - f|}(t) = 0$$

$$f_n \xrightarrow{\mathbf{I}_S} f \quad \text{iff} \ \lim_{n\to\infty} \mathbf{I}_S(m, |f_n - f|) = 0$$

Theorem (relationship between convergence in measure and in mean)

Let $(m, f) \in \mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ and $S \in \mathfrak{S}$ with no zero divisors. Then the following assertions are equivalent:

(i) $f_n \xrightarrow{\mathbf{I}_S} f$;

(ii) $f_n \xrightarrow{m} f$.

$$f_n \xrightarrow{\text{s-}m} f \quad \text{iff} \quad \lim_{n\to\infty} m(\{x \in X; |f_n(x) - f(x)| > 0\}) = 0$$

Example: Let $X = [0,1]$, λ be the Lebesgue measure on $\mathscr{B}(X)$ and take the sequence of functions

$$f_n(x) = \frac{1}{n} \qquad \text{and} \qquad f(x) = 0$$

for $x \in X$. Then $f_n \overset{\text{s-}\lambda}{\not\to} f$, but

$$\lim_{n\to\infty} \mathbf{I}_S(\lambda, |f_n - f|) = \lim_{n\to\infty} \mathbf{I}_S(\lambda, f_n) = \mathbf{I}_S(\lambda, f) = 0,$$

i.e. $f_n \xrightarrow{\mathbf{I}_S} f$.

Open problem: For which class of semicopulas (of measures, eventually) is strict convergence in measure equivalent to mean convergence?

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Theorem (convergence in measure)

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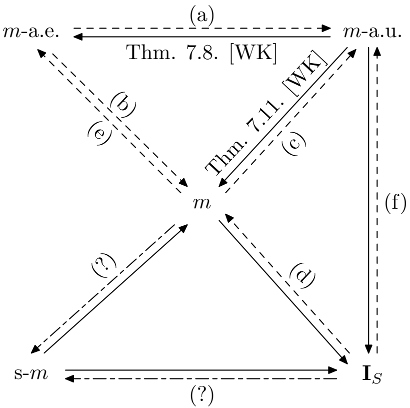
(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \xrightarrow{\text{s-}m} f$, it holds $\lim_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.

Theorem (convergence in mean)

Let $S \in \mathfrak{S}$ be continuous without zero divisors and $m \in \mathscr{M}^1_{(X,\mathscr{A})}$. Then the following assertions are equivalent:

(i) m is autocontinuous;

(ii) for all $f, (f_n)_1^\infty \in \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}$ such that $f_n \xrightarrow{\mathbf{I}_S} f$, it holds $\lim_{n\to\infty} \mathbf{I}_S(m, f_n) = \mathbf{I}_S(m, f)$.



Relationships among convergences schematically

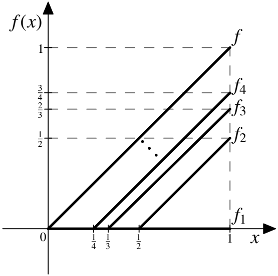
- for a copula C define a mapping
$\mathbf{K}_C : \bigcup_{(X,\mathscr{A})\in\mathscr{S}} \left(\mathscr{M}^1_{(X,\mathscr{A})} \times \mathscr{F}^{[0,1]}_{(X,\mathscr{A})}\right) \to [0,1]$ as

$$\mathbf{K}_C(m,f) := P_C\left(\{(x,y)\in[0,1]^2; y \le h_{m,f}(x)\}\right)$$

Example: Let $X=[0,1]$, λ be the Lebesgue measure on $\mathscr{B}(X)$. Consider the monotone sequence of functions

$$f_n(x) = \max\left\{0, x-\frac{1}{n}\right\}$$

for $n\in\mathbb{N}$, $x\in X$ and $f(x)=x$ on X.



Then

$$\mathbf{K}_W(\lambda, f_n) = 0, \text{ but } \mathbf{K}_W(\lambda, f) = 1.$$

Thank you for your attention!