Convergence theorems for (semi)copula - based universal integrals

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A $[0, 1]$-valued universal integral is a functional

$$I : \bigcup_{(X, A) \in \mathcal{S}} \left( \mathcal{M}_1^1(X, A) \times \mathcal{F}^{[0,1]}(X, A) \right) \to [0, 1]$$

satisfying the following conditions:

1) for all $(X, A) \in \mathcal{S}$, $m_1, m_2 \in \mathcal{M}_1^1(X, A)$ and $f_1, f_2 \in \mathcal{F}^{[0,1]}(X, A)$ with $m_1 \leq m_2$, $f_1 \leq f_2$ we have $I(m_1, f_1) \leq I(m_2, f_2)$;

2) for all $(X, A) \in \mathcal{S}$, $m \in \mathcal{M}_1^1(X, A)$ and $A \in \mathcal{A}$ we have $m(A) = I(m, 1_A)$;

3) for all $(X, A) \in \mathcal{S}$, $m \in \mathcal{M}_1^1(X, A)$ and $c \in [0, 1]$ we have $I(m, c \cdot 1_X) = c$;

4) $I(m_1, f_1) = I(m_2, f_2)$ for all integral equivalent pairs.
A $[0, 1]\text{-valued universal integral}$ is a functional

$$I : \bigcup_{(X, \mathcal{A}) \in \mathcal{I}} \left( \mathcal{M}_1^{(X, \mathcal{A})} \times \mathcal{F}^{[0,1]}(X, \mathcal{A}) \right) \to [0, 1]$$

satisfying the following conditions:

1. (I1) for all $(X, \mathcal{A}) \in \mathcal{I}$, $m_1, m_2 \in \mathcal{M}_1^{(X, \mathcal{A})}$ and $f_1, f_2 \in \mathcal{F}^{[0,1]}(X, \mathcal{A})$ with $m_1 \leq m_2$, $f_1 \leq f_2$ we have $I(m_1, f_1) \leq I(m_2, f_2)$;

2. (I2) for all $(X, \mathcal{A}) \in \mathcal{I}$, $m \in \mathcal{M}_1^{(X, \mathcal{A})}$ and $A \in \mathcal{A}$ we have $m(A) = I(m, 1_A)$;

3. (I3) for all $(X, \mathcal{A}) \in \mathcal{I}$, $m \in \mathcal{M}_1^{(X, \mathcal{A})}$ and $c \in [0, 1]$ we have $I(m, c \cdot 1_X) = c$;

4. (I4) $I(m_1, f_1) = I(m_2, f_2)$ for all integral equivalent pairs.
A $[0, 1]$-valued universal integral is a functional

$$I : \bigcup_{(X, \mathcal{A}) \in \mathcal{I}} \left( \mathcal{M}^1_{(X, \mathcal{A})} \times \mathcal{F}^{[0, 1]}_{(X, \mathcal{A})} \right) \rightarrow [0, 1]$$

satisfying the following conditions:

1. for all $(X, \mathcal{A}) \in \mathcal{I}, m_1, m_2 \in \mathcal{M}^1_{(X, \mathcal{A})}$ and $f_1, f_2 \in \mathcal{F}^{[0, 1]}_{(X, \mathcal{A})}$ with $m_1 \leq m_2, f_1 \leq f_2$ we have $I(m_1, f_1) \leq I(m_2, f_2)$;

2. for all $(X, \mathcal{A}) \in \mathcal{I}, m \in \mathcal{M}^1_{(X, \mathcal{A})}$ and $A \in \mathcal{A}$ we have $m(A) = I(m, 1_A)$;

3. for all $(X, \mathcal{A}) \in \mathcal{I}, m \in \mathcal{M}^1_{(X, \mathcal{A})}$ and $c \in [0, 1]$ we have $I(m, c \cdot 1_X) = c$;

4. $I(m_1, f_1) = I(m_2, f_2)$ for all integral equivalent pairs.
A [0, 1]-valued universal integral is a functional

\[ I : \bigcup_{(X, \mathcal{A}) \in \mathcal{I}} \left( \mathcal{M}^1_{(X, \mathcal{A})} \times \mathcal{F}^{[0,1]}_{(X, \mathcal{A})} \right) \rightarrow [0, 1] \]

satisfying the following conditions:

I1) for all \((X, \mathcal{A}) \in \mathcal{I}, m_1, m_2 \in \mathcal{M}^1_{(X, \mathcal{A})}\) and \(f_1, f_2 \in \mathcal{F}^{[0,1]}_{(X, \mathcal{A})}\) with \(m_1 \leq m_2, f_1 \leq f_2\) we have \(I(m_1, f_1) \leq I(m_2, f_2)\);

I2) for all \((X, \mathcal{A}) \in \mathcal{I}, m \in \mathcal{M}^1_{(X, \mathcal{A})}\) and \(A \in \mathcal{A}\) we have \(m(A) = I(m, 1_A)\);

I3) for all \((X, \mathcal{A}) \in \mathcal{I}, m \in \mathcal{M}^1_{(X, \mathcal{A})}\) and \(c \in [0, 1]\) we have \(I(m, c \cdot 1_X) = c\);

I4) \(I(m_1, f_1) = I(m_2, f_2)\) for all integral equivalent pairs.
A $[0, 1]$-valued universal integral is a functional

$$I : \bigcup_{(X, \mathcal{A}) \in \mathcal{I}} \left( M^1_{(X, \mathcal{A})} \times F_{[0, 1]}(X, \mathcal{A}) \right) \rightarrow [0, 1]$$

satisfying the following conditions:

I1) for all $(X, \mathcal{A}) \in \mathcal{I}$, $m_1, m_2 \in M^1_{(X, \mathcal{A})}$ and $f_1, f_2 \in F_{[0, 1]}(X, \mathcal{A})$ with $m_1 \leq m_2$, $f_1 \leq f_2$ we have $I(m_1, f_1) \leq I(m_2, f_2);$  

I2) for all $(X, \mathcal{A}) \in \mathcal{I}$, $m \in M^1_{(X, \mathcal{A})}$ and $A \in \mathcal{A}$ we have $m(A) = I(m, 1_A);$  

I3) for all $(X, \mathcal{A}) \in \mathcal{I}$, $m \in M^1_{(X, \mathcal{A})}$ and $c \in [0, 1]$ we have $I(m, c \cdot 1_X) = c;$  

I4) $I(m_1, f_1) = I(m_2, f_2)$ for all integral equivalent pairs.
for \((m, f) \in \mathcal{M}_1(X, \mathcal{A}) \times \mathcal{F}^{[0,1]}(X, \mathcal{A})\) we may introduce a single function 
\(h_{m,f} : [0,1] \to [0,1]\) as follows 
\[ h_{m,f}(t) := m(\{x \in X ; f(x) \geq t\}) \]

For \((m, f) \in \mathcal{M}_1(X, \mathcal{A}) \times \mathcal{F}^{[0,1]}(X, \mathcal{A})\) the smallest \([0,1]-valued\) universal integral having \(S\) as the underlying semicopula is given by 
\[ I_S(m, f) := \sup_{t \in [0,1]} S(t, h_{m,f}(t)). \]
for \((m, f) \in \mathcal{M}^1(X, \mathcal{A}) \times \mathcal{F}^{[0,1]}(X, \mathcal{A})\) we may introduce a single function \(h_{m,f} : [0, 1] \to [0, 1]\) as follows

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h_{m,f}(t) := m(\{x \in X; f(x) \geq t\})
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For \((m, f) \in \mathcal{M}^1(X, \mathcal{A}) \times \mathcal{F}^{[0,1]}(X, \mathcal{A})\) the smallest \([0, 1]\)-valued universal integral having \(S\) as the underlying semicopula is given by

\[
I_S(m, f) := \sup_{t \in [0,1]} S(t, h_{m,f}(t)).
\]

\(I_M\) ... the Sugeno integral
\(I_\Pi\) ... the Shilkret integral
\(I_T\) ... the Sugeno-Weber integral (with \(T\) being a fixed strict t-norm)
Theorem (monotone convergence I)

Let $S \in \mathcal{G}$ be left-continuous and $m \in \mathcal{M}^1_1$. Then the following assertions are equivalent:

(i) $m$ is continuous from below;

(ii) for all $f$, $(f_n)_n \in \mathcal{F}([0,1])$ such that $f_n \nearrow f$, it holds

$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$

Example: Consider $X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ and

$$m(A) = \begin{cases} 
0, & A = \emptyset \\
1, & \text{else,}
\end{cases}$$

$$f_n(x) = \begin{cases} 
0, & x \in \left[\frac{1}{n}, 1\right] \\
1, & \text{else}
\end{cases}$$

for $n \in \mathbb{N}$ and $f(x) = 0$ on $X$. For every $n \in \mathbb{N}$ and $t \in [0,1]$ we have

$$h_{m,f_n}(t) = m\left([0, \frac{1}{n}]\right) = 1.$$ However,

$$I_S(m, f_n) = \sup_{t \in [0,1]} S(t, h_{m,f_n}(t)) = \sup_{t \in [0,1]} S(t, 1) = 1 \text{ and } I_S(m, f) = 0.$$
Theorem (monotone convergence II)

Let $S \in \mathcal{G}$ be right-continuous and $m \in \mathcal{M}^{1}_{(X, \mathcal{A})}$. Then the following assertions are equivalent:

(i) $m$ is continuous from above;

(ii) for all $f, (f_{n})_{n=1}^{\infty} \in \mathcal{F}_{(X, \mathcal{A})}^{[0,1]}$ such that $f_{n} \searrow f$, it holds \[ \lim_{n \to \infty} I_{S}(m, f_{n}) = I_{S}(m, f). \]

Example: Consider $X = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1])$ and

\[
m(A) = \begin{cases} 
0, & A = \emptyset \\
1, & \text{else,}
\end{cases} \\
f_{n}(x) = \begin{cases} 
0, & x \in ]\frac{1}{n}, 1] \\
1, & \text{else}
\end{cases}
\]

for $n \in \mathbb{N}$ and $f(x) = 0$ on $X$. For every $n \in \mathbb{N}$ and $t \in [0, 1]$ we have $h_{m,f_{n}}(t) = m([0, \frac{1}{n}]) = 1$. However, \[ I_{S}(m, f_{n}) = \sup_{t \in [0,1]} S(t, h_{m,f_{n}}(t)) = \sup_{t \in [0,1]} S(t, 1) = 1 \text{ and } I_{S}(m, f) = 0. \]
Theorem (everywhere convergence)  

Let $S \in \mathcal{S}$ be continuous and $m \in \mathcal{M}(\mathcal{X}, \mathcal{A})$. Then the following assertions are equivalent:

(i) $m$ is continuous;

(ii) for all $f, (f_n) \in \mathcal{F}^{[0,1]}(\mathcal{X}, \mathcal{A})$ such that $f_n \to f$, it holds

$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$

Example: Consider $X = ]0, 1]$, $\mathcal{A} = \mathcal{B}(]0, 1])$ and

$$m(A) = \begin{cases} 0, & A = \emptyset \\ 1, & \text{else,} \end{cases}$$

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for $n \in \mathbb{N}$ and $f(x) = 0$ on $X$. For every $n \in \mathbb{N}$ and $t \in [0, 1]$ we have

$h_{m,f_n}(t) = m([0, \frac{1}{n}]) = 1$. However,

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Theorem (everywhere convergence)

Let $S \in \mathcal{S}$ be continuous and $m \in \mathcal{M}^1(X, \mathcal{A})$. Then the following assertions are equivalent:

(i) $m$ is continuous;

(ii) for all $f, (f_n)_n \in \mathcal{F}^{[0,1]}(X, \mathcal{A})$ such that $f_n \to f$, it holds
$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$

**Example:** Consider $X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ and

$$m(A) = \begin{cases} 0, & A = \emptyset \\ 1, & \text{else,} \end{cases}$$

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for $n \in \mathbb{N}$ and $f(x) = 0$ on $X$. For every $n \in \mathbb{N}$ and $t \in [0,1]$ we have

$$h_{m, f_n}(t) = m\left( [0, \frac{1}{n}] \right) = 1.$$ However,

$$I_S(m, f_n) = \sup_{t \in [0,1]} S(t, h_{m, f_n}(t)) = \sup_{t \in [0,1]} S(t, 1) = 1$$ and

$$I_S(m, f) = 0.$$
Theorem (everywhere convergence)

Let $S \in \mathcal{S}$ be continuous and $m \in \mathcal{M}_{(X, \mathcal{A})}^1$. Then the following assertions are equivalent:

(i) $m$ is continuous;

(ii) for all $f, (f_n)_1^\infty \in \mathcal{F}_{(X, \mathcal{A})}^{[0,1]}$ such that $f_n \to f$, it holds

$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$

\[ f_n \overset{m-a.e.}{\longrightarrow} f \text{ iff } (\exists A \in \mathcal{A}, \ m(A) = 0) \ f_n \to f \text{ on } X \setminus A \]

Theorem (almost everywhere convergence)

Let $S \in \mathcal{S}$ be continuous and $m \in \mathcal{M}_{(X, \mathcal{A})}^1$. Then the following assertions are equivalent:

(i) $m$ is null-additive and continuous;

(ii) for all $f, (f_n)_1^\infty \in \mathcal{F}_{(X, \mathcal{A})}^{[0,1]}$ such that $f_n \overset{m-a.e.}{\longrightarrow} f$, it holds

$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$
Convergence theorems for integral $I_S$

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Introduction and motivation

Monotone convergence

Almost uniform convergence

Convergence in measure and mean convergence

$C$-universal integral

Theorem (almost uniform convergence)

Let $S \in \mathcal{S}$ be continuous and $m \in \mathcal{M}_1(X, \mathcal{A})$. Then the following assertions are equivalent:

(i) $m$ is monotone autocontinuous;

(ii) for all $f, (f_n)_{\infty} \in \mathcal{F}^{[0,1]}(X, \mathcal{A})$ such that $f_n \xrightarrow{m-a.u.} f$, it holds $\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f)$. 

Proof:

- $m : \mathcal{A} \to [0,1]$ is said to be autocontinuous from above, iff $\lim_{n \to \infty} m(A \cup B_n) = m(A)$ for all $A \in \mathcal{A}$ and $(B_n)_{\infty} \in \mathcal{A}$ with $\lim_{n \to \infty} m(B_n) = 0$.

- $m : \mathcal{A} \to [0,1]$ is said to be autocontinuous from below, iff $\lim_{n \to \infty} m(A \setminus B_n) = m(A)$ for all $A, (B_n)_{\infty} \in \mathcal{A}$ with $\lim_{n \to \infty} m(B_n) = 0$. 

\[ f_n \xrightarrow{m-a.u.} f \text{ iff } (\forall \varepsilon \in [0, 1]) \ (\exists A_\varepsilon \in \mathcal{A}, \ m(A_\varepsilon) < \varepsilon) \ f_n \xrightarrow{u.} f \text{ on } X \setminus A_\varepsilon \]
\[ f_n \xrightarrow{m} f \quad \text{iff} \quad (\forall t \in ]0, 1[) \lim_{n \to \infty} h_m, |f_n - f|(t) = 0 \]

\[ f_n \xrightarrow{I_S} f \quad \text{iff} \quad \lim_{n \to \infty} I_S(m, |f_n - f|) = 0 \]

**Theorem (relationship between convergence in measure and in mean)**

Let \((m, f) \in \mathcal{M}^1_1(X, \mathcal{A}) \times \mathcal{F}([0,1], X, \mathcal{A})\) and \(S \in \mathfrak{S}\) with no zero divisors. Then the following assertions are equivalent:

(i) \(f_n \xrightarrow{I_S} f\);

(ii) \(f_n \xrightarrow{m} f\).
\[ f_n \xrightarrow{s-m} f \quad \text{iff} \quad \lim_{n \to \infty} m(\{x \in X; |f_n(x) - f(x)| > 0\}) = 0 \]

**Example:** Let \( X = [0, 1] \), \( \lambda \) be the Lebesgue measure on \( \mathcal{B}(X) \) and take the sequence of functions

\[ f_n(x) = \frac{1}{n} \quad \text{and} \quad f(x) = 0 \]

for \( x \in X \). Then \( f_n \xrightarrow{s-\lambda} f \), but

\[ \lim_{n \to \infty} I_S(\lambda, |f_n - f|) = \lim_{n \to \infty} I_S(\lambda, f_n) = I_S(\lambda, f) = 0, \]

i.e. \( f_n \xrightarrow{I_S} f \).

**Open problem:** For which class of semicopulas (of measures, eventually) is strict convergence in measure equivalent to mean convergence?
\[ f_n \xrightarrow{s-m} f \text{ iff } \lim_{n \to \infty} m(\{x \in X; |f_n(x) - f(x)| > 0\}) = 0 \]

**Example:** Let \( X = [0, 1] \), \( \lambda \) be the Lebesgue measure on \( \mathcal{B}(X) \) and take the sequence of functions

\[ f_n(x) = \frac{1}{n} \quad \text{and} \quad f(x) = 0 \]

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\[ \lim_{n \to \infty} I_s(\lambda, |f_n - f|) = \lim_{n \to \infty} I_s(\lambda, f_n) = I_s(\lambda, f) = 0, \]

i.e. \( f_n \xrightarrow{I_s} f \).

**Open problem:** For which class of semicopulas (of measures, eventually) is strict convergence in measure equivalent to mean convergence?
**Theorem (convergence in measure)**

Let $S \in \mathcal{S}$ be continuous and $m \in M^1(X, \mathcal{A})$. Then the following assertions are equivalent:

(i) $m$ is autocontinuous;

(ii) for all $f, (f_n)_{n=1}^{\infty} \in \mathcal{F}_{[0,1]}(X, \mathcal{A})$ such that $f_n \overset{s-m}{\rightarrow} f$, it holds

$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$

**Theorem (convergence in mean)**

Let $S \in \mathcal{S}$ be continuous without zero divisors and $m \in M^1(X, \mathcal{A})$. Then the following assertions are equivalent:

(i) $m$ is autocontinuous;

(ii) for all $f, (f_n)_{n=1}^{\infty} \in \mathcal{F}_{[0,1]}(X, \mathcal{A})$ such that $f_n \overset{I_S}{\rightarrow} f$, it holds

$$\lim_{n \to \infty} I_S(m, f_n) = I_S(m, f).$$
Relationships among convergences schematically
for a copula $C$ define a mapping
\[
K_C : \bigcup_{(X, \mathcal{A}) \in \mathcal{S}} \left( \mathcal{M}^X_{\mathcal{A}} \times \mathcal{F}^{[0,1]}_{\mathcal{A}} \right) \to [0, 1] \text{ as }
\]
\[
K_C(m, f) := P_C \left( \{(x, y) \in [0, 1]^2; y \leq h_{m,f}(x)\} \right)
\]

Example: Let $X = [0, 1]$, $\lambda$ be the Lebesgue measure on $\mathcal{B}(X)$. Consider the monotone sequence of functions
\[
f_n(x) = \max \left\{ 0, x - \frac{1}{n} \right\}
\]
for $n \in \mathbb{N}$, $x \in X$ and $f(x) = x$ on $X$.

Then
\[
K_W(\lambda, f_n) = 0, \quad \text{but} \quad K_W(\lambda, f) = 1.
\]
Thank you for your attention!