Integrals based on copulas

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Universal integral

E.P. Klement, R. Mesiar and E. Pap: *A universal integral as common frame for Choquet and Sugeno integral.*

A mapping $I : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}\right) \to [0,\infty]$,

- $\mathcal{S}$ is the class of all measurable spaces,
- $\mathcal{M}^{(X,\mathcal{A})}$ is the set of all monotone measures $m : \mathcal{A} \to [0,\infty]$
- $\mathcal{F}^{(X,\mathcal{A})}$ is the set of all $\mathcal{A}$-measurable functions $f : X \to [0,\infty]$,

such that:

(i) I is increasing in both coordinates,

(ii) $I(m, c\,1_A)$ depends on c and $m(A)$ only, independently of $(X, \mathcal{A})$, $m \in \mathcal{M}^{(X,\mathcal{A})}$ and $A \in \mathcal{A}$,

(iii)

$$I(m_1, f_1) = I(m_2, f_2)$$

for all couples $(m_1, f_1) \in \mathcal{M}^{(X_1,\mathcal{A}_1)} \times \mathcal{F}^{(X_1,\mathcal{A}_1)}$, $(m_2, f_2) \in \mathcal{M}^{(X_2,\mathcal{A}_2)} \times \mathcal{F}^{(X_2,\mathcal{A}_2)}$ such that

$$m_1(f_1 \geq t) = m_2(f_2 \geq t)$$

for all $t \in]0, \infty[$.

is a universal integral.

For a special case of (fuzzy) universal integrals on $[0,1]$ we deal with

- $f \in \mathcal{F}^{(X,\mathcal{A})}$ such that $\mathrm{Ran}\, f \subseteq [0,1]$,
- $m \in \mathcal{M}^{(X,\mathcal{A})}$ such that $m(X) = 1$

it should hold

(iv) $I(m, 1_A) = m(A),\ m \in \mathcal{M}^{(X,\mathcal{A})},\ A \in \mathcal{A}$,

(v) $I(m, c\,1_X) = c,\ m \in \mathcal{M}^{(X,\mathcal{A})}, c \in [0,1]$.

Then necessarily

$$I(m, c\,1_A) = S(c, m(A)),$$

where $S : [0,1]^2 \to [0,1]$ is a semicopula, i.e., a monotone two-place function with neutral element 1.

We recall three typical universal integrals on $[0,1]$:

- **Choquet integral** ,

$$Ch(m,f) = \int_0^1 m(f \geq t)\,dt;$$

- **Sugeno integral**,

$$Su(m,f) = \sup\{\min\{t, m(f \geq t)\} \mid t \in [0,1]\};$$

- **Shilkret integral**,

$$Sh(m,f) = \sup\{t \cdot m(f \geq t) \mid t \in [0,1]\}.$$

Decomposition Integral

Y. Even and E. Lehrer: *Decomposition-Integral: Unifying Choquet and the Concave Integrals*

Let $\mathcal{H}$ be a set of some set systems from $\mathcal{A}$, where $(X, \mathcal{A})$ is a fixed measurable space. Then the decomposition integral $D_{\mathcal{H}} : \mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})} \to [0, \infty]$ is given by

$$D_{\mathcal{H}}(m, f) = \sup \left\{ \sum_{i \in J} a_i \cdot m(A_i) \,\middle|\, (A_i)_{i \in J} \in \mathcal{H}, \right.$$
$$\left. a_i \geq 0,\ i \in J,\ \sum_{i \in J} a_i 1_{A_i} \leq f \right\}.$$

Two distinguished decomposition integrals:

Shilkret integral $\mathcal{H} = \{\{A\} | A \in \mathcal{A}\}$,

Choquet integral $\mathcal{H} = \{(A_i)_{i \in J}$ a finite chain in $\mathcal{A}\}$.

Observe that each decomposition integral is positively homogeneous, i.e.,

$$D_{\mathcal{H}}(m, c\,f) = cD_{\mathcal{H}}(m, f) \text{ for all } c \in]0, \infty[.$$

Evidently, a decomposition integral which is also a universal integral on $[0,1]$ is linked to the standard product Π as the underlying semicopula. Denote as

$$\mathcal{M}_{(1)}^{(X,\mathcal{A})} = \{m \in \mathcal{M}^{(X,\mathcal{A})} \mid m(X) = 1\} \text{ and}$$
$$\mathcal{F}_{[0,1]}^{(X,\mathcal{A})} = \{f \in \mathcal{F}^{(X,\mathcal{A})} \mid \mathrm{Ran}\, f \subseteq [0,1]\}.$$

R. Mesiar and A. Stupňanová: *Decomposition integrals.*

The characterization of all decomposition integrals which are also universal integrals on $[0,1]$:
The mappings
$I^{(n)}: \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(1)}^{(X,\mathcal{A})} \times \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}\right) \to [0,\infty],\ n \in \mathbb{N}$, given by

$$I^{(n)}(m,f) = \sup\left\{\sum_{i=1}^{n} a_i \cdot m(A_i) \,|\, a_1,\ldots,a_n \geq 0, \right.$$
$$\left. \{A_1,\ldots,A_n\} \in \mathcal{A} \text{ is a chain}, \sum_{i=1}^{n} a_i\, 1_{A_i} \leq f \right\}, \quad (1)$$

together with the Choquet integral.

$$Sh = I^{(1)} \leq I^{(2)} \leq \cdots \leq I^{(n)} \leq \ldots,$$

$$\sup\{I^{(n)} | n \in \mathbb{N}\} = Ch,$$

and if $\mathrm{card}\, X = n$, then

$$I^{(n)} = Ch$$

on $(X, \mathcal{A}) \equiv (X, 2^X)$.

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A copula $C : [0,1]^2 \to [0,1]$ is a supermodular semicopula, i.e., a semicopula satisfying

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$.

- Π *independence copula*
- $M = \min$ *comonotone dependence copula*
- $W(x,y) = \max\{0, x+y-1\}$ *countermonotone dependence copula*
- $H(x,y) = \frac{xy}{x+y-xy}$ if $(x,y) \neq (0,0)$ *Ali - Mikhail - Haq copula* (also called the Hamacher product)

$$W < \Pi < H < M$$

Note also that each copula C is related to a probability measure $P_C : \mathcal{B}([0,1]^2) \to [0,1]$ constrained by

$$P_C([0,x] \times [0,y]) = C(x,y),\ (x,y) \in [0,1]^2.$$

In KMP for a given copula $C : [0,1]^2 \to [0,1]$ two universal integrals on $[0,1]$ were introduced:

- $I_{(C)} : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(1)}^{(X,\mathcal{A})} \times \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}\right) \to [0,\infty]$ given by

$$I_{(C)}(m,f) = \sup\{C\,(t, m(f \geq t)) \mid t \in [0,1]\}$$

- $I_C : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(1)}^{(X,\mathcal{A})} \times \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}\right) \to [0,\infty]$ given by

$$I_C(m,f) = P_C\left(\{(x,y) \in [0,1]^2 \,|\, y \leq m(f \geq x)\}\right).$$

- $I_{(C)}$ is the weakest universal integral linked to C as the underlying semicopula
- $$I_{\Pi} = Ch$$
- $$I_{(M)} = I_M = Su$$

Looking on the product as the copula Π, we can rewrite the formula (1) into

$$I^{(n)}(m,f) = \sup \left\{ \sum_{i=1}^{n} \left(\Pi \left(\sum_{j=1}^{i} a_j, m(f \geq \sum_{j=1}^{i} a_j) \right) - \Pi \left(\sum_{j=1}^{i-1} a_j, m(f \geq \sum_{j=1}^{i} a_j) \right) \right) \mid a_1, \dots, a_n \geq 0 \right\}, \qquad (2)$$

with convention $\sum_{j=1}^{0} a_j = 0$.

Definition

Let $n \in \mathbb{N}$ and a copula $C : [0,1]^2 \to [0,1]$ be fixed. The (n, C)-universal integral on [0, 1]
$I_C^{(n)} : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(1)}^{(X,\mathcal{A})} \times \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}\right) \to [0,1]$ is given by

$$I_C^{(n)}(m,f) = \sup\left\{\sum_{i=1}^{n}\left(C\left(\sum_{j=1}^{i} a_j, m(f \geq \sum_{j=1}^{i} a_j)\right) - C\left(\sum_{j=1}^{i-1} a_j, m(f \geq \sum_{j=1}^{i} a_j)\right)\right) \mid a_1, \ldots, a_n \geq 0\right\}. \qquad (3)$$

Remark

Note that if $n = 1$, for arbitrary semicopula $S : [0,1]^2 \to [0,1]$ the functional $I_S^{(1)} = I_{(S)}$ given by (3) is the (weakest) universal integral linked to S. However, $I_S^{(2)}$ does not satisfy the axiom (i) of universal integrals (i.e., the monotonicity), in general. To ensure this, S should be supermodular, i.e., a copula.

Example

Let $X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, $m : \mathcal{B}([0,1]) \to [0,1]$ be the standard Lebesque measure, and let $f, g : X \to [0,1]$ be given by $g = \frac{1}{2} \cdot 1_{[0,\frac{1}{2}]}$, $f = \frac{1}{4} \cdot (1_{[0,\frac{1}{2}]} + 1_{[0,\frac{1}{4}]})$. Then

$$I_S^{(2)}(g,m) = S\left(\frac{1}{2}, \frac{1}{2}\right) \text{ and}$$

$$I_S^{(2)}(f,m) = S\left(\frac{1}{2}, \frac{1}{4}\right) + \sup\left\{S\left(a, \frac{1}{2}\right) - S\left(a, \frac{1}{4}\right) \mid a \in \left[0, \frac{1}{4}\right]\right\}.$$

Define a semicopula $S : [0,1]^2 \to [0,1]$ by

$$S(x,y) = \begin{cases} 0 & \text{if } x+y < \frac{3}{4} \\ \frac{1}{4} & \text{if } x+y \geq \frac{3}{4},\ x \leq \frac{1}{2},\ y \leq \frac{1}{2} \\ \min\{x,y\} & \text{elsewhere.} \end{cases}$$

Then

$$I_S^{(2)}(g,m) = \frac{1}{4} < \frac{1}{2} = I_S^{(2)}(f,m),$$

but $f \leq g$, violating the monotonicity of $I_S^{(2)}$. Observe that S is not a copula.

For any fixed copula C

$$I_{(C)} = I_C^{(1)} \leq I_C^{(2)} \leq \cdots \leq I_C^{(n)} \leq \cdots \leq I_C.$$

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- $C = \Pi$, it holds $I_{\Pi}^{(n)} = I^{(n)}, \ n \in \mathbb{N}$
- $C = M$

$$I_{(M)} = I_M^{(1)} = I_M^{(2)} = \cdots = I_M^{(n)} = \cdots = Su.$$

For any copula C, one can introduce a universal integral

$$I_{[C]} = \sup\{I_C^{(n)} \mid n \in \mathbb{N}\},$$

and obviously $I_{[C]} \leq I_C$.

- $I_{[\Pi]} = \sup\{I_{\Pi}^{(n)} \mid n \in \mathbb{N}\} = I_{\Pi} = Ch$
- $I_{[M]} = \sup\{I_{M}^{(n)} \mid n \in \mathbb{N}\} = I_{M} = Su$
- $I_{[H]} = I_H$
- $$I_{[W]}(m,f) = \sup\{I_W^{(n)}(m,f) \mid n \in \mathbb{N}\} = \\ = P_W(\{(x,y) \in [0,1]^2 \mid y < m(f \geq x)\}).$$

Note that $I_{[W]}$ is also a universal integral related to W, and that $I_{[W]} < I_W$.

Example

Let $X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, $m : \mathcal{B}([0,1]) \to [0,1]$ be the standard Lebesque measure, and let $f : X \to [0,1]$ be the identity function, $f(x) = x$.

Then

$$Ch(m,f) = \int_0^1 (1-t)dt = \frac{1}{2},$$

$$Sh(m,f) = I_\Pi^{(1)}(m,f) = \sup\{t(1-t) \mid t \in [0,1]\} = \frac{1}{4},$$

$$I_\Pi^{(2)}(m,f) = \sup\{a(1-a) + b(1-a-b) \mid a,b,a+b \in [0,1]\} = \frac{1}{3},$$

$$I_\Pi^{(n)}(m,f) = \frac{n}{2(n+1)}.$$

$$\sup\{\tfrac{n}{2(n+1)} \mid n \in \mathbb{N}\} = \tfrac{1}{2} = Ch(m, f).$$

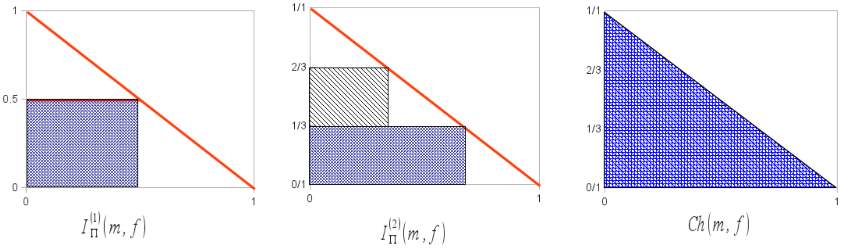

Figure 1. Shaded areas correspond to integrals $I_\Pi^{(1)}$, $I_\Pi^{(2)}$ and Ch.

Moreover, $Su(m, f) = \frac{1}{2}$, see Figure 2. Observe that $Su(m, f)$ is related to a rectangle $[0, a] \times [0, m(f \geq a)]$ containing the biggest part of the main diagonal, which in our case is just the square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ containing one half of the main diagonal.

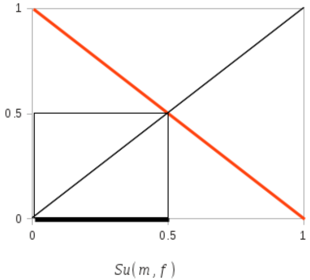

Figure 2. Sugeno integral.

Further,

$$I_W^{(n)}(m,f) = I_{[W]}(m,f) = 0,\ n \in \mathbb{N},$$

while

$$I_W(m,f) = 1.$$

Hence $I_{[W]}(m,f) < I_W(m,f)$.

For the copula H, we have the next results:

$$I_H^{(1)}(m,f) = \frac{1}{3},$$

$$I_H^{(n)}(m,f) = \sup\left\{\sum_{i=1}^{n}(H(b_i, 1-b_i) - H(b_{i-1}, 1-b_i)) \middle| \right.$$

$$\left. 0 = b_0 \leq b_1 \leq \cdots \leq b_n \leq 1 \right\},$$ see Table

Table: Values of integrals $I_H^{(n)}(m, f)$.

n	$I_H^{(n)}(m, f)$
1	0.3333
2	0.3821
3	0.4048
4	0.4182
5	0.4271
6	0.4335
7	0.4383
∞	0.4728

$$I_H(m, f) = \int_0^1 \left(\int_0^{1-x} \frac{2xy}{(x+y-xy)^3} dy \right) dx = \frac{4\sqrt{3}\pi - 9}{27}$$

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For any copula $C : [0,1]^2 \to [0,1]$, we have introduced a hierarchical set of universal integrals

$$I_{(C)} = I_C^{(1)} \leq I_C^{(2)} \leq \cdots \leq I_C^{(n)} \leq \cdots \leq I_C,$$

and its limit member

$$I_C^{(\infty)} = I_{[C]} \leq I_C.$$

As we have shown, we cannot ensure the equality $I_{[C]} = I_C$, in general. This observation leads to open problem of characterizing all copulas C for which $I_{[C]} = I_C$.

We expect applications of our functionals in multicriteria decision area and optimization tasks. Note that copulas can express the dependence between scores and weights of criteria in multicriteria decision area. As a particular application one can expect proposals of new bibliometric indices (observe that several of them are related to the Sugeno integral, e.g. the famous *h*-index).