Introduction

Diagonal sections of n-dimensional copulas

2 dimensional copula with an a priori given diagonal section

n-dimensional diagonal copula with an a priori given diagonal section

Examples

Concluding remarks

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**Definition 1.**

For an n-dimensional copula

\[ C : [0, 1]^n \to [0, 1], \ n \geq 2, \]

its diagonal section \( \delta_C(x) \) is defined by

\[ \delta_C(x) = C(x, \ldots, x). \]

We will discuss the reverse problem, i.e., how to find for an a priori given diagonal section \( \delta : [0, 1]^n \to [0, 1] \) (of some unknown copula) an n-dimensional copula \( C : [0, 1]^n \to [0, 1] \) so that \( \delta = \delta_C \).
Let for a fixed $n \in \{2, 3, \ldots \}$, $C_n$ be the class of all $n$-dimensional copulas and $D_n$ be the class of all diagonal sections of copulas from $C_n$.

If the function $d : [0, 1] \rightarrow [0, 1]$ is an element of $D_n$ then it satisfies the next conditions:

(D1) $d$ is non-decreasing,
(D2) $d \leq id_{[0,1]}$,
(D3) $d(1) = 1$,
(D4) $d$ is $n$-Lipschitz, i.e., $|d(x) - d(y)| \leq n|x - y|$ for all $x, y \in [0, 1]$. 
Proposition 1.

Let \( d : [0, 1] \rightarrow [0, 1] \) be a function and \( n \in \{2, 3, \ldots\} \) be a fixed dimension. Then \( d \) is a diagonal section of some \( n \)-dimensional copula, i.e., \( d \in D_n \) if and only if \( d \) satisfies conditions (D1) - (D4).
Copulas $M(x, y), W(x, y), \Pi(x, y)$ and their diagonal sections $\delta_M, \delta_W$ and $\delta_\Pi$
The classes $C_n$ and $D_n$ are convex.

$D_n$ is closed under suprema (infima).

The smallest element of $D_n$ is given by

$$d_n^-(x) = \max(0, nx - n + 1),$$

while its greatest element is given by $d_n^+(x) = x$.

The class $C_n$ is not closed under suprema (infima).

The greatest element of $C_n$ is the comonotonicity copula $M$, $M(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$.

The smallest element in $C_n$, $n > 2$ does not exist.

In the case of $C_2$, the smallest element is the countermonotonicity copula $W$, $W(x_1, x_2) = \max(0, x_1 + x_2 - 1)$. 
The smallest and the greatest elements of $D_n$ for $n = 2$ and $n = 3$
Bertino copulas

For any $d \in D_2$, the function $B_d : [0, 1]^2 \rightarrow [0, 1]$ given by

$$B_d (x, y) = \bigvee_{t \in [x \land y, x \lor y]} \left( d(t) - (t - x)^+ - (t - y)^+ \right)^+,$$

where $u^+ = \max (u, 0)$ for $u \in \mathbb{R}$, is a copula. $B_d$ is the smallest copula with diagonal section $d$, and it is simultaneously the smallest quasi-copula possessing diagonal section $d$.

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Diagonal copulas

For any $d \in \mathcal{D}_2$, the function $K_d : [0, 1]^2 \rightarrow [0, 1]$ given by

$$K_d(x, y) = \min \left( x, y, \frac{d(x) + d(y)}{2} \right)$$

(2)

is a copula.

$K_d$ is the greatest symmetric copula with diagonal section $d$, but not necessarily the greatest one.

\(^a\) (Fredricks, G.A., Nelsen, R.B., 1997)
Copula \( K_d(x, y) \) for \( d(x) = x^2 \)
There are several other constructions of a copula with an a priori given diagonal section $d$, however, these methods are not universal, they can be applied to diagonal sections from some special subdomains of $D_2$.

This is, for example, the case of semilinear copulas, biconic copulas, or the construction methods based on patchwork techniques.
Proposition 2.

Let $A$ and $B$ be symmetric copulas from $C_2$ with the same diagonal section $d \in D_2$. Then the function $C_{A,B} : [0,1]^2 \rightarrow [0,1]$ given by

$$C_{A,B} (x,y) = \begin{cases} A (x,y) & \text{if } x \leq y, \\ B (x,y) & \text{else}, \end{cases}$$

is a copula from $C_2$, and $d_{C_{A,B}} = d_A = d_B = d$.

The proposition allow to introduce for any $d \in D_2$ two copulas $C_{B_d,K_d}$ and $C_{K_d,B_d}$ with diagonal section $d$. For any $d \in D_2$, $d \neq d^+$, $\text{card}\{B_d, K_d, C_{B_d,K_d}, C_{K_d,B_d}\} = 4$. 
n-dimensional diagonal copulas

Proposition 3.
For a fixed $n \in \{2, 3, \ldots\}$, let $d \in D_n$. Then the function

$$J_d : [0, 1]^n \rightarrow [0, 1]$$

given by

$$J_d (x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \min (f (x_{i+1}), \ldots, f (x_{i+n-1}), d (x_{i+n}))$$

(4)

where $f : [0, 1] \rightarrow [0, 1]$ is given by

$$f (x) = \frac{nx - d (x)}{n - 1}$$

and $x_j = x_{j-n}$ for $j \in \{n + 1, \ldots, 2n\}$, is a copula, $J_d \in C_n$. 

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DIAGONAL COPULAS AND QUASI-COPULAS
For $n = 2$, $f(x) = 2x - d(x)$, and

$$J_d(x_1, x_2) = \frac{1}{2} (\min (2x_2 - d(x_2), d(x_1)) + \min (2x_1 - d(x_1), d(x_2))) =$$

$$= \min \left( x_1, x_2, \frac{d(x_1) + d(x_2)}{2} \right) = K_d(x_1, x_2),$$

i.e., copula introduced by Jaworski coincide with diagonal copula $K_d$. 
The generalization of Bertino copula $B_d$ for $n > 2$, $d \in D_n$ is not a universal method for n-dimensional copulas. $B_d^{-}(x_1, x_2, ..., x_n) = W(x_1, x_2, ..., x_n) = \max (0, \sum_{i=1}^{n} x_i - (n - 1))$ is not a copula.

Similarly, the generalization of diagonal copulas $K_d$ for fixed $d \in D_n$, $n > 2$, given by

$$K_d(x_1, x_2, ..., x_n) = \min \left( x_1, x_2, ..., x_n, \frac{d(x_1) + ... + d(x_n)}{n} \right)$$

is not a universal method for n-dimensional copulas. $K_d$ is a symmetric quasi-copula for any $d \in D_n$. 
Due to ordinal sum representation of copulas, we can introduce a notion of the ordinal sums of diagonal sections,

\[ d = \left( \langle a_k, b_k, d_k \rangle | k \in K \right), \]

where \( K \) is an index system, \( \langle a_k, b_k \rangle_{k \in K} \) is a disjoint system of open subintervals of \([0, 1]\), and \( d_k \in D_n \) for each \( k \in K \). Then

\[
d : [0, 1] \rightarrow [0, 1] \text{ is given by }
\]

\[
d (x) = \begin{cases} 
a_k + (b_k - a_k) d_k \left( \frac{x-a_k}{b_k-a_k} \right) & \text{if } x \in ]a_k, b_k[ \text{ for some } k \in K, \\
x & \text{else.}
\end{cases}
\]
The corresponding function $f : [0, 1] \rightarrow [0, 1]$ given by

$$f(x) = \frac{nx - d(x)}{n - 1}$$

can be written in the form

$$f(x) = \begin{cases} a_k + (b_k - a_k) f_k \left( \frac{x-a_k}{b_k-a_k} \right) & \text{if } x \in ]a_k, b_k[, \\ x & \text{else.} \end{cases}$$
Proposition 4.
For a fixed $n \in \{2, 3, \ldots\}$, let $d \in \mathcal{D}_n$ be an ordinal sum,

$$d = (\langle a_k, b_k, d_k \rangle | k \in \mathcal{K}).$$

Then $J_d$ is an ordinal sum copula $J_d = (\langle a_k, b_k, J_{d_k} \rangle | k \in \mathcal{K}).$

Construction (4) and ordinal sum constructions commute, construction (4) does not commute with convex sums construction. The only elements of $\mathcal{D}_n$ which do not admit a non-trivial convex sum decomposition are the ordinal sums of type $(\langle a_k, b_k, d^- \rangle | k \in \mathcal{K}).$ We denote their class by $\mathcal{E}_n.$
Proposition 5.

For a fixed \( n \in \{2, 3, \ldots\} \), let \( d \in \mathcal{D}_n \setminus \mathcal{E}_n \), i.e.,

\[
d = \lambda d_1 + (1 - \lambda) d_2
\]

for some \( d_1, d_2 \in \mathcal{D}_n, \ d_1 \neq d_2, \ \lambda \in ]0, 1[ \). Then

\[
J_{\lambda, d_1, d_2} = \lambda J_{d_1} + (1 - \lambda) J_{d_2}
\]

is a copula from \( \mathcal{C}_n \) with diagonal section \( d \), and \( J_{\lambda, d_1, d_2} \neq J_d \), in general.
For $n = 2$, any construction of a binary copula from an a priori given diagonal section $d \in D_2$ can be “dualized”, using the notion of a survival diagonal section.
**Example 1.** Consider the weakest diagonal section \( d^- \in D_3 \). Then \( J_{d^-} \) and \( K_{d^-} \) are described in Table 1.

<table>
<thead>
<tr>
<th>domain</th>
<th>( J_{d^-} )</th>
<th>( K_{d^-} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, \frac{2}{3}]^3)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([\frac{2}{3}, 1]^3)</td>
<td>( x_1 + x_2 + x_3 - 2 )</td>
<td>( x_1 + x_2 + x_3 - 2 )</td>
</tr>
<tr>
<td>([0, \frac{2}{3}] \times [0, \frac{2}{3}] \times [\frac{2}{3}, 1])</td>
<td>( \min \left( \frac{x_1}{2}, \frac{x_2}{2}, x_3 - \frac{2}{3} \right) )</td>
<td>( \min \left( \frac{x_1}{2}, x_2 - \frac{2}{3}, x_3 - \frac{2}{3} \right) )</td>
</tr>
<tr>
<td>([0, \frac{2}{3}] \times [\frac{2}{3}, 1] \times [0, \frac{2}{3}])</td>
<td>( \min \left( \frac{x_1}{2}, x_2 - \frac{2}{3}, \frac{x_3}{2} \right) )</td>
<td>( \min \left( \frac{x_1}{2}, x_2, x_3 - \frac{2}{3} \right) )</td>
</tr>
<tr>
<td>([\frac{2}{3}, 1] \times [0, \frac{2}{3}] \times [0, \frac{2}{3}])</td>
<td>( \min \left( \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2} \right) )</td>
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<td>([0, \frac{2}{3}] \times [\frac{2}{3}, 1] \times [\frac{2}{3}, 1])</td>
<td>( \min \left( \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2} \right) + \min \left( \frac{x_1}{2}, x_3 - \frac{2}{3} \right) )</td>
<td>( \min \left( \frac{x_1}{2}, x_2 + x_3 - \frac{2}{3} \right) )</td>
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<td>([\frac{2}{3}, 1] \times [0, \frac{2}{3}] \times [\frac{2}{3}, 1])</td>
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<td>( \min \left( \frac{x_1}{2}, x_2 + x_3 - \frac{2}{3} \right) )</td>
</tr>
</tbody>
</table>
\( J_{d^-} \leq K_{d^-} \).

\( J_{d^-} \) is singular copula from \( C_3 \).

Its support consists of 3 segments connecting the point \( \left( \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right) \) with vertices \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\) and the mass 1 is uniformly distributed over the support of \( J_{d^-} \).

The proper quasi-copula \( K_{d^-} \) has a negative mass \(-\frac{1}{3}\) on each of rectangles

\[
\left[ 0, \frac{1}{3} \right] \times \left[ \frac{2}{3}, 1 \right] \times \left[ \frac{2}{3}, 1 \right], \left[ \frac{2}{3}, 1 \right] \times \left[ 0, \frac{1}{3} \right] \times \left[ \frac{2}{3}, 1 \right]
\]

and

\[
\left[ \frac{2}{3}, 1 \right] \times \left[ \frac{2}{3}, 1 \right] \times \left[ 0, \frac{1}{3} \right].
\]
Example 2. For the product copula $\Pi \in C_n, n \geq 2$, the corresponding diagonal section $d \in D_n$ is given by $d_\Pi(x) = x^n$. For $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq 1$, it holds

$$J_{d_\Pi}(x_1, \ldots, x_n) = \frac{1}{n} \left( x_1^n + \sum_{i=2}^{n} \min \left( \frac{nx_1 - x^n}{n-1}, x_i^n \right) \right).$$

Consider diagonal sections $d_1, d_2 \in D_3$ given by

$$d_1(x) = \begin{cases} 
0 & \text{if } x \leq \frac{1}{4}, \\
\frac{x}{2} - \frac{1}{8} & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4}, \\
3x - 2 & \text{else.}
\end{cases}$$

and

$$d_2(x) = \begin{cases} 
2x^3 & \text{if } x \leq \frac{1}{4}, \\
2x^3 - \frac{x}{2} + \frac{1}{8} & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4}, \\
2x^3 - 3x + 2 & \text{else.}
\end{cases}$$
Then \( \frac{d_1 + d_2}{2} = d_\Pi \) and thus the copula \( \frac{1}{2} (J_{d_1} + J_{d_2}) \) has \( d_\Pi \) as its diagonal section.

Diagonal sections \( d_1, d_2 \) and \( d_\Pi \).
Concluding remarks

- We have opened the problem of constructing n-dimensional copulas with a predescribed diagonal section, with the stress on higher dimensions, i.e., \( n \in \{3, 4, \ldots\} \).

- Though there are some similarities with well developed case \( n = 2 \), several techniques cannot be used for higher dimensions.

- Especially, there is no universal construction leading to a smallest copula having a given diagonal section (for \( n > 2 \), there is no smallest copula in \( C_n \)).

- We aim to focus on extension of particular methods known for the case \( n = 2 \), starting from a diagonal section \( d \in D_n \) with some specific properties, such as semilinear copulas or biconic copulas in the 2-dimensional case.
Thanks for attention
Introduzione  

Diagonal sections of n-dimensional copulas  


