EXTREME VALUE ANALYSIS BASED ON COPULAS AND THEIR DIAGONALS

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The problem of risk accumulation

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- environmental risks (earthquakes, flood);
- civil engineering (the problem of water at flood level in Venice).
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Generali collaboration: Solvency II and the dependence between extreme events


Gaussian copula
Traditional, but underestimation of the risk of joint downside movements.

Gumbel copula
Overestimation of the risk.

t-Student copula
Dependence in the tails, but not in the center.
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Elliptic copulas

The Normal copula is the dependence function

$$C_n^\Phi(u; \Omega) = \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n); \Omega),$$  \hspace{1cm} (1)

where \( \Phi_n \) is the cdf for the \( n \)-variate standard normal distribution with correlation matrix \( \Omega \).

The t-Student copula

$$C_n^\Psi(u; \Omega, \nu) = \Psi_n(\Psi^{-1}(u_1, \nu), \ldots, \Psi^{-1}(u_n; \nu); \Omega, \nu),$$  \hspace{1cm} (2)

where \( \Phi_n \) denotes the cdf of an \( n \)-variate Student's t distribution with correlation matrix \( \Omega \) and degrees of freedom parameter \( \nu > 2 \).
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Archimedean copula

Gumbel copula: an extreme copula

\[
C^G_n(u; a) = \exp \left( - \left( \sum_{i=1}^{n} (- \log u_i)^a \right)^{\frac{1}{a}} \right),
\]

with \( a \geq 1 \), where \( a = 1 \) implies independence. Upper tail dependence but lower tail independence.
Extreme copula

Definition: MEV copulas

An extreme copula satisfies

\[ C(u_1^t, \ldots, u_n^t, \ldots, u_N^t) = C^t(u_1, \ldots, u_n, \ldots, u_N) \quad \forall t > 0. \]

MEV copulas are easily recognized from

\[ A(x) = -\log G(x), \]

being homogeneous of order 1, i.e., \( A(tx) = tA(x) \), for all \( t > 0 \), with \( \tilde{G}(x) = C(e^{-x_1}, \ldots, e^{-x_m}). \)

Remark

Gaussian copula is not an extreme value copula.
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Gaussian copula is **not** an extreme value copula.
The set $C$ of 2-copulas is compact with any of the following topologies, equivalent on $C$: punctual convergence, uniform convergence on $[0, 1]^2$, weak convergence of the associated probability measure.

Let $E_x(C)$ be the set of the extreme points of $C$. Then Choquet’s representation of $C$ similar to the Birkhoff’s theorem:

$$C \text{ is the convex hull of } E_x(C).$$
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$\mathcal{C}$ is the convex hull of $\mathcal{E}_x(\mathcal{C})$. 
The determination of the extreme points of $C$ is an open problem.

**Theorem**

Any element of $C$ that possesses a left or right inverse is extreme.

**Examples**

Ordinal sums of $C^-(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ and $C^+(u_1, u_2) = \min(u_1, u_2)$ are extreme points of $C$. 
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Link between extreme value copulas and the multivariate extreme value theory

Denote $\chi_{n,m}^+ = \max(X_{n,1}, \ldots, X_{n,k}, \ldots, X_{n,m})$ with \{X_{n,k}\}, k i.i.d. random variables with the same distribution. Let $G_n$ be the marginal distribution of the univariate extreme $\chi_{n,m}^+$. Then, the joint limit distribution $G$ of $(\chi_{1,m}^+, \ldots, \chi_{n,m}^+, \ldots, \chi_{N,m}^+)$ is such that

$$G(\chi_1^+, \ldots, \chi_n^+, \ldots, \chi_N^+) = C(G_1(\chi_1^+), \ldots, G_n(\chi_n^+), \ldots, G_N(\chi_N^+)),$$

where $C$ is an extreme value copula and $G_n$ a non-degenerate univariate extreme value distribution.
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Let us first consider $m$ independent random variables $X_1, \ldots, X_k, \ldots, X_m$ with the same probability function $F$. The distribution of the extremes $\chi_m^+ = (\bigwedge_{k=1}^m X_k)$ is also given by Fisher-Tippett theorem:

If there exist some constants $a_m$ and $b_m$ and a non-degenerate limit distribution $G$ such that

$$\lim_{m \to \infty} P \left\{ \frac{\chi_m^+ - b_m}{a_m} \leq x \right\} = G(x) \quad \forall x \in \mathbb{R}$$

then $G$ is one of the following distributions:
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Distributions

Fréchet

\[ G(x) = \Upsilon_\alpha(x) = \begin{cases} 
0 & x \leq 0 \\
\exp(-x^{-\alpha}) & x > 0
\end{cases} \]

Weibull

\[ G(x) = \Psi_\alpha(x) = \begin{cases} 
\exp(-(-x^\alpha)) & x \leq 0 \\
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Gumbel

\[ G(x) = \Lambda(x) = \exp(-e^{-x}) \quad x \in \mathbb{R} \]

In this case, we say that \( F \) belongs to the maximum domain of attraction of \( G \), \( F \in MDA(G) \).
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Max-stable distribution

**Definition**

A non-degenerate rv $X$ (the corresponding distribution or df) is called *max-stable* if it satisfies the identity in law

$$\max(X_1, \ldots, X_n) \overset{d}{=} c_n X + d_n$$

for i.i.d. $X, X_1, \ldots, X_n$, appropriate constants $c_n > 0$, $d_n \in \mathbb{R}$ and every $n \geq 2$.

**Proposition**

The class of multivariate extreme value distributions is the class of max-stable distribution functions with nondegenerate marginals.
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The class of multivariate extreme value distributions is the class of max-stable distribution functions with nondegenerate marginals.
Our problem

Generic choice of copulas also depending on Kendall’s $\tau$ (which is also in the $a$ parameter of the Gumbel through the link $a = \frac{1}{1-\tau}$). Therefore, we have the following situation:

$$\tau_C = 4 \int \int_{I^2} C(u, v) dC(u, v) - 1 =$$

$$= Arch.Cop. \ 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt$$
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My proposal: Diagonals

Let $C : [0, 1] \to [0, 1]$ be an $n$-dimensional copula, $n \geq 2$. The function

$$\delta : [0, 1] \to [0, 1], \delta(t) = C(t, \ldots, t)$$

is called a **diagonal section** or **diagonal** for short.

Kendall’s $\tau$ in connection with the general copula $C$.

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Tail dependence

Upper tail dependence

If a bivariate copula $C$ is such that

$$\lim_{u \to 1} \frac{\bar{C}(u, u)}{1 - u} = \lambda_U$$

exists, then $C$ has upper tail dependence for $\lambda_U \in (0, 1]$ and no upper tail dependence for $\lambda_U = 0.$

Lower tail dependence

If a bivariate copula $C$ is such that

$$\lim_{u \to 0} \frac{C(u, u)}{u} = \lambda_L$$

exists, then $C$ has lower tail dependence for $\lambda_L \in (0, 1]$ and no lower tail dependence for $\lambda_L = 0.$
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Tail dependence
Tail dependence: some examples

**Gumbel family**

The Gumbel family has upper tail dependence, with

\[ \lambda_U = 2 - 2^{\frac{1}{\alpha}} \]

**Clayton family**

The Clayton family has lower tail dependence for \( \alpha > 0 \), since

\[ \lambda_L = 2^{-\frac{1}{\alpha}} \]

**Frank family**

The Frank family has neither lower nor upper tail dependence.
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$$\delta_C(t) = C(t, t)$$

$\lambda_U$ in connection with the general copula $C$ by:

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t, t)}{1 - t} = 2 - \delta'_C(1^-).$$

$$\lambda_L = \lim_{t \to 0^+} \frac{\delta(t)}{t}$$

Remark

The measure $\lambda$ is extensively used in extreme value theory. It is the probability that one variable is extreme given that the other is extreme.
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**Remark**

The measure $\lambda$ is extensively used in extreme value theory. It is the probability that one variable is extreme given that the other is extreme.
Let $U = (U_1, \ldots, U_n)$ be an $n$-variate random variable with uniform margins, $U_i \sim U(0, 1)$, $C$ its distribution function (hence a copula) and $\delta$ the diagonal section of $C$. Then $\delta$ is a distribution function of the random variable $\max\{U_1, \ldots, U_n\} = U_{n:n}$. 
Application to two assets option pricing

Let $Q_n$ and $Q$ be the risk-neutral probability distributions of $S_n(T)$ and $S(T) = (S_1(T) \ldots S_N(T))^\top$. With arbitrage theory, we can show that

$$Q(\infty, \ldots, \infty, S_n(T), \infty, \ldots, \infty) = Q_n(S_n(T)).$$

$\Rightarrow$ The margins of $Q$ are RNDs $Q_n$ of Vanilla options.

Remark

European option prices permit to characterize the probability distribution of $S_n(T)$

$$\Phi(T, K) := Q_n(K).$$
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Bivariate case

For a call max option $\Phi(T, K)$ is the diagonal section of the copula

$$\Phi(T, K) = C(Q_1(K), Q_2(K))$$

For a spread option, we have

$$\Phi(T, K) = \int_0^{+\infty} \partial_1 C(Q_1(x), Q_2(x + K)) dQ_1(x).$$
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Concluding remarks

OPEN PROBLEMS:

1. The parametric form of both bivariate and multivariate copulas is not well tractable;

2. Current multivariate extreme value theory, from an applied point of view, only allows for a treatment of fairly low-dimensional problems.
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THANK YOU FOR YOUR ATTENTION!