A topology on residuated lattices

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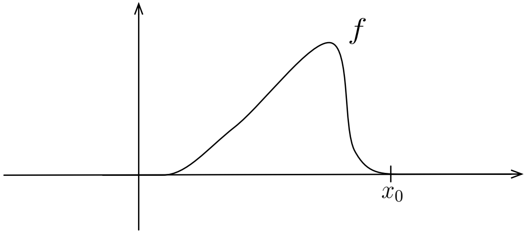
Continuity of fuzzy sets

Is this fuzzy set continuous?



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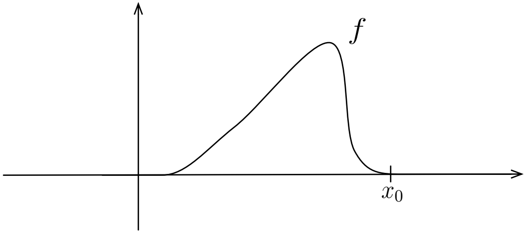

But doesn't continuity depend on t-norm?

$\otimes$: a left-continuous t-norm, $\rightarrow$: its residuum.

Biresiduum: $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$, the degree of similarity of a and b.

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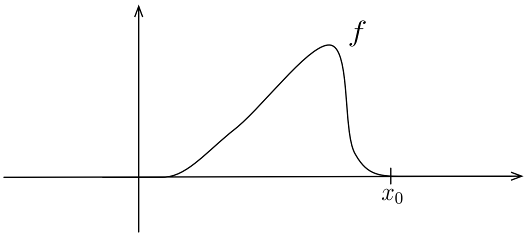

But doesn't continuity depend on t-norm?

$\otimes$: a left-continuous t-norm, $\to$: its residuum.

Biresiduum: $a \leftrightarrow b = (a \to b) \wedge (b \to a)$, the degree of similarity of a and b.

Example

If $\otimes$ is the product t-norm, then for each $x > 0$, $x \leftrightarrow 0 = 0$.
Why should then $\lim \frac{1}{n} = 0$?
And for the above fuzzy set, if $x_n \to x_0$ from the left, should $f(x_n) \to f(x_0)$?

Our main goal

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Convergence and continuity depends on topology. Our main goal is:

Define a topology on $[0, 1]$ (and, more general, on any residuated lattice) which would take into account proximity given by $\leftrightarrow$

Topology

Definition (topology)

A *topology* on a set X is a system τ of subsets of X such that

1. $\emptyset, X \in \tau$,
2. if $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$,
3. if $U, V \in \tau$, then $U \cap V \in \tau$.

Elements of τ: *open sets*.

Definition (basis of topology)

A *basis of topology* τ is a system $\sigma \subseteq \tau$ such that for each open set U, $x \in U$ there is a $V \in \sigma$ s.t. $x \in V \subseteq U$.

A system σ of subsets of X is a basis of some topology on X iff it is a covering of X and for each $V_1, V_2 \in \sigma$ and $x \in V_1 \cap V_2$ there exists $W \in \sigma$ such that $x \in W \subseteq V_1 \cap V_2$.

Continuous mappings

Definition (continuous mappings)

Let $f\colon X \to Y$ be a mapping of topological spaces. f is continuous if for each open set $V \subseteq Y$ the set $f^{-1}(V) \subseteq X$ is open.

Residuated lattices

Definition

A *residuated lattice*: algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice
2. $\langle L, \otimes, 1\rangle$ is a commutative monoid
3. $\otimes$ and $\to$ satisfy adjointness property: $a \otimes b \leq c$ iff $a \leq b \to c$.

Three important examples on $[0, 1]$

$\mathbf{L} = \langle [0,1], \min, \max, \otimes, \to, 0, 1\rangle$ given by left-continuous (continuous) $\otimes$.

- Łukasiewicz: $a \otimes b = \max(a + b - 1, 0)$, $a \to b = \min(1 - a + b, 1)$.
- Gödel (minimum): $a \otimes b = \min(a, b)$, $a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$
- Goguen (product): $a \otimes b = a \cdot b$, $a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$

Open ball topology

Let $\mathbf{L}$ be a residuated lattice.

Definition (radius)

$r \in L$ is a *radius*, if $r \vee s = 1$ implies $s = 1$.

Definition (relation $\prec$)

a is *totally smaller than* b, if $b \to a$ is a radius. We write $a \prec b$, or $b \succ a$.

Definition (open ball)

The open ball with center a and radius r: $B^r_{\mathbf{L}}(a) = \{x \in L \mid (x \leftrightarrow a) \succ r\}$.

Theorem

The system of all open balls in $\mathbf{L}$ forms a basis of a topology on L.

The topology will be denoted $\tau_{\mathbf{L}}$ and called *the open ball topology*.

If $\mathbf{L}$ is linearly ordered

- Each $r < 1$ is a radius,
- $\prec = <$,
- open balls are intervals: $|r \otimes a, r \to a|$.
- Thus, $\tau_{\mathbf{L}}$ is always stronger than the order topology (natural topology in the case of $[0, 1]$).
- Thus, some $\mathbf{L}$-sets, continuous in the usual sense, are not continuous w.r.t. $\tau_{\mathbf{L}}$.

Continuity of residuated lattice operations

Definition

continuity condition $\mathbf{L}$ is said to *satisfy the continuity condition*, if for each radius r there is a radius s such that $a \succ s$ implies $a \otimes a \succ r$.

Theorem

If $\mathbf{L}$ *satisfies the continuity condition, then all the operations* $\wedge, \vee, \otimes, \rightarrow$ *are continuous w.r.t.* $\tau_{\mathbf{L}}$*.*

If $\mathbf{L}$ is linearly ordered

- $\mathbf{L}$ satisfies the continuity condition.
- Thus, all the operations $\wedge, \vee, \otimes, \rightarrow$ are continuous w.r.t. $\tau_{\mathbf{L}}$.
- Even if $\otimes$ or $\rightarrow$ is not continuous in the usual sense.

Further remarks

- Extensional fuzzy sets are continuous.
- A generalization to $\mathbf{L}$-similarity spaces is possible: if $\approx$ is an $\mathbf{L}$-equivalence on a set X, then sets

$$B_X^r(x) = \{y \in X \mid y \approx x \succ r\},\ x \in X, r \text{ is a radius,}$$

  form a basis of a topology on X.
- De Baets, Mesiar (1999): continuous Archimedean t-norms define a metric on $[0, 1]$. The topology of this metric is exactly the open ball topology.

Some things to do

- investigate properties of $\tau_{\mathbf{L}}$
- connection to uniformity
- find applications!
- (and more. . .)