On Suborbital Graphs for A Special Congruence Subgroup

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Outline

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Modular Group

\[ T : z \to \frac{az+b}{cz+d} : a, b, c, d \text{ are real numbers, } ad - bc = 1 \]

- \[ PSL(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} \]

- \[ \Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \rightarrow \text{Modular Group} \]
\[ \Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a \equiv d \equiv 1 \pmod{n}, b \equiv c \equiv 0 \pmod{n} \right\} \]

\[ \Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a \equiv d \equiv 1 \pmod{n}, c \equiv 0 \pmod{n} \right\} \]

\[ \Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; c \equiv 0 \pmod{n} \right\} \]

\[ \Gamma(n) \leq \Gamma_1(n) \leq \Gamma_0(n) \leq \Gamma \leq PSL(2, \mathbb{R}) \]

\[ \Gamma(n) \triangleleft \Gamma, \Gamma(n) \triangleleft \Gamma_0(n), \Gamma(n) \triangleleft \Gamma_1(n), \Gamma_1(n) \triangleleft \Gamma_0(n) \]
THE MODULAR GROUP AND
GENERALIZED FAREY GRAPHS

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1. Introduction

The modular group

\[ \Gamma = \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{ \pm I \} \]

is the quotient of the unimodular group \( \text{SL}(2, \mathbb{Z}) \) by its centre \( \{ \pm I \} \). Thus the elements of \( \Gamma \) are the pairs of matrices

\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \quad (a, b, c, d \in \mathbb{Z}, \ ad - bc = 1); \]

(1.1)

we will omit the symbol \( \pm \) and identify each matrix with its negative.

It is both traditional and useful to represent \( \Gamma \) as a group of Möbius transformations of the upper half-plane

\[ \mathcal{U} = \{ z \in \mathbb{C} : \text{Im} \ z > 0 \}, \]

with the elements (1.1) acting by

\[ z \mapsto \frac{az + b}{cz + d}. \]

(1.2)

For example, using the fact that this action of \( \Gamma \) is discontinuous, one can show that \( \Gamma \) is isomorphic to a free product \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \), more specifically, \( \Gamma \) is generated by the elements

\[ X = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad Y = \left( \begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right). \]

(1.3)

with defining relations...
This paper is organized as follows:

- $\Gamma$ acts transitively but imprimitively on $\hat{Q}$
- They summarize Sims’ theory
- They obtained a graph $G_{u,n}$ on which $\Gamma$ acts
- They examine the Farey graph $(G_{1,1})$ as a simplest case
- They focus a subgraph $F_{u,n}$
- They found edge and circuit conditions respectively
- They give a conjecture that $G_{u,n}$ is forest iff it contains no triangles

Jones, G.A., Singerman D. and Wicks, K.
The modular group and generalized Farey graphs.
Jones, Singerman, and Wicks used the notion of the imprimitive action for a $\Gamma$-invariant equivalence relation induced on $\hat{\mathbb{Q}}$ by the congruence subgroup $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{n} \right\}$ to obtain some suborbital graphs and examined their connectedness and forest properties. They left the forest problem as a conjecture, which was settled down by Akbaş.

We introduce a different $\Gamma$-invariant equivalence relation by using the congruence subgroup $\Gamma_0^3(n)$ instead of $\Gamma_0(n)$ and obtain some results for the newly constructed subgraphs $F_{u,n}^3$. 
Some papers concerning suborbitals graphs

Akbaş, M.
On suborbital graphs for the modular group.

Keskin, R.
Suborbital graphs for some Hecke groups.
Let \((G, \Delta)\) be transitive permutation group. 
\(G\) acts on \(\Delta \times \Delta\) by \((g \in G; \alpha, \beta \in \Delta)\)

\[ g(\alpha, \beta) = (g(\alpha), g(\beta)) \]

* The orbits of this action are called **suborbitals** of \(G\)
* The orbit containing \((\alpha, \beta)\) is denoted by \(O(\alpha, \beta)\)
From \(O(\alpha, \beta)\) we can form a **suborbital graph** \(G(\alpha, \beta)\)

**Sims, C.C.**
Graphs and finite permutation groups.
Math.Z., 95 (1967), 76-86.
its vertices are the elements of \( \Delta \)

there is a directed edge from \( \gamma \) to \( \delta \) if \((\gamma, \delta) \in O(\alpha, \beta)\) means that

\[
\gamma \to \delta : \iff T \in G \text{ such that } T(\alpha) = \delta, T(\beta) = \gamma
\]

If \( \alpha = \beta \), the corresponding suborbital graph \( G(\alpha, \beta) \), called the trivial suborbital graph, is self-paired: it consist of a loop based at each vertex \( \alpha \in \Delta \).
By a directed circuit of length $m$ (or a closed edge path) we mean a sequence $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_m \rightarrow v_1$ of different vertices where $m \geq 3$; an anti-directed circuit will denote a configuration like the above with at least an arrow (not all) reversed.

- If $m = 2$, the circuit $v_1 \rightarrow v_2 \rightarrow v_1$ will be called a $2$–gon.
- If $m = 3$ or $4$ then the circuit is called a triangle or rectangle.

A graph is called a forest if it contains no circuits other than 2-gons.
Background

Our study

Kesicioğlu, Akbaş

On Suborbital Graphs for A Special Congruence Subgroup
Imprimitive Action

Let us give a general discussion of primitivity of permutation groups. Let \((G, \Delta)\) be a transitive permutation group, consisting of a group \(G\) acting on a set \(\Delta\) transitively. An equivalence relation \(\approx\) on \(\Delta\) is called \(G\)-invariant if, whenever \(\alpha, \beta \in \Delta\) satisfy \(\alpha \approx \beta\), then \(g(\alpha) \approx g(\beta)\) for all \(g \in G\).

The equivalence classes are called blocks, and the block containing \(\alpha\) is denoted \([\alpha]\).

We call \((G, \Delta)\) imprimitive if \(\Delta\) admits some \(G\)-invariant equivalence relation different from

- the identity relation, \(\alpha \approx \beta\) iff \(\alpha = \beta\).
- the universal relation, \(\alpha \approx \beta\) for all \(\alpha, \beta \in \Delta\).

Otherwise \((G, \Delta)\) is called primitive. These two relations are supposed to be trivial relations.
Theorem

Let \((G, \Delta)\) be a transitive permutation group. \((G, \Delta)\) is primitive if and only if \(G_\alpha\), the stabilizer of \(\alpha \in \Delta\), is a maximal subgroup of \(G\) for each \(\alpha \in \Delta\).

* We suppose that \(G_\alpha < H < G\). Since \(G\) acts transitively, for \(g, h \in G\)

\[
g(\alpha) \approx h(\beta) \text{ if and only if } g^{-1}h \in H
\]

is an imprimitive \(G\)-invariant equivalence relation.

Bigg, N.L.; White, A.T.,
Permutation groups and combinatorial structures.
London Mathematical Society Lecture Note Series, 33 (1979)
Subgraph $F_{u,n}^3$

We apply these ideas to the case:

$$G_\alpha = \Gamma_\infty^3, H = \Gamma_0^3(n), G = \Gamma^3$$

$$\Delta = \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$$

$$\Gamma^3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ab + cd \equiv 0 (mod 3) \right\}$$

$$\Gamma_0^3(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^3 : c \equiv 0 (mod n) \right\}$$

$$\Gamma^3_\infty = \langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \rangle$$

$$\Delta = \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \rightarrow \text{extended rational numbers}$$

$$\Gamma^3_\infty < \Gamma_0^3(n) \leq \Gamma^3$$
\( \Gamma^3 \) must be one of the three types

\[
\begin{pmatrix}
3a & b \\
c & 3d
\end{pmatrix},
\begin{pmatrix}
a & 3b \\
3c & d
\end{pmatrix},
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

where \( a, b, c, \) and \( d, \) are rational integers and \( a, b, c, d \neq 0(\text{mod}3) \) in the third matrix.

**Lemma**

*The action of \( \Gamma^3 \) on \( \hat{\mathbb{Q}} \) is transitive.*
We get the following imprimitive $\Gamma^3$- invariant equivalence relation on $\hat{\mathbb{Q}}$ by $\Gamma^3_0(n)$ as
\[
\frac{r}{s} \approx \frac{x}{y} \quad \text{if and only if} \quad g^{-1} h \in \Gamma^3_0(n)
\]
where $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ and $h$ is similar.

From the above we can easily verify that
\[
\frac{r}{s} \approx \frac{x}{y} \quad \text{if and only if} \quad ry - sx \equiv 0 \pmod{n}.
\]

The equivalence classes are called blocks and the block containing $\frac{x}{y}$ is denoted by $[\frac{x}{y}]$. 
Since $\Gamma$ acts transitively on $\hat{\mathbb{Q}}$, every suborbital $O(\alpha, \beta)$ contains a pair $(\infty, \frac{u}{n})$ for $\frac{u}{n} \in \hat{\mathbb{Q}}$, $n \geq 0$, $(u, n) = 1$. In this case, we denote the suborbital graph by $G_{u,n}$ for short.

As $\Gamma$ permutes the blocks transitively, all subgraphs corresponding to blocks are isomorphic. Therefore we will only consider the subgraph $F_{u,n}^3$ of $G_{u,n}$ whose vertices form the block $[\infty] = [\frac{1}{0}]$, which is the set

$$\{ \frac{x}{y} \in \hat{\mathbb{Q}} | y \equiv 0 \pmod{n} \}.$$
Edge Conditions

Theorem

\[
\frac{r}{s} \to \frac{x}{y} \text{ in } F_{u,n}^3 \text{ if and only if}
\]

1. If \( r \equiv 0 (\text{mod} 3) \), then \( x \equiv \pm ur (\text{mod} n) \), \( y \equiv \pm us (\text{mod} 3n) \) and \( ry - sx = \pm n \) or,

2. If \( s \equiv 0 (\text{mod} 3) \), then \( x \equiv \pm ur (\text{mod} 3n) \), \( y \equiv \pm us (\text{mod} n) \) and \( ry - sx = \pm n \) or,

3. If \( r, s \not\equiv 0 (\text{mod} 3) \), then \( x \equiv \pm ur (\text{mod} n) \), \( x \not\equiv \pm ur (\text{mod} 3n) \), \( y \equiv \pm us (\text{mod} n) \), \( y \not\equiv \pm us (\text{mod} 3n) \) and \( ry - sx = \pm n \).
Theorem
\[ \Gamma^3_0(n) \text{ permutes the vertices and edges of } F^3_{u,n} \text{ transitive.} \]

Theorem
\[ F^3_{u,n} \text{ contains no directed triangles.} \]

Theorem
\[ F^3_{u,n} \text{ contains a 2-gon if and only if } n \not\equiv 0 (\text{mod} 3) \text{ and } u^2 \equiv -1 (\text{mod} n). \]
Connectedness

**Definition**

A subgraph $K$ of $F^{3}_{u,n}$ is called connected if any pair of its vertices can be joined by a path in $K$. 

![Diagram of connected subgraph](image)
Background

Our study

Theorem

The subgraph $F_{0,1}^3$ is not connected.

Since $\infty \to \frac{0}{1}$ is an edge in $F_{0,1}^3$ and $F_{0,1}^3$ is periodic with period 3, we can do calculation only, in the strip $\frac{0}{1} \leq \text{Rez} \leq \frac{3}{1}$. It is clear that $\infty$ is adjacent to $\frac{0}{1}$ and $\frac{3}{1}$ in $F_{0,1}^3$, but to no intermediate vertices. We shall show that the edge $\frac{1}{1} \to \frac{2}{1} \in F_{0,1}^3$ is not adjacent to $\infty$. We assume that $\frac{1}{1}$ can be joined to $\infty$ by a path $D$ in $F_{0,1}^3$. We may assume that $D$ has the form

\[ \frac{r}{s} \rightarrow \frac{y}{x} \in F_{0,1} \Leftrightarrow \]

1) $r \equiv 0 \ (mod \ 3)$ ise $y \equiv 0 (mod \ 3)$ ve $ry - sx = \pm 1$

2) $s \equiv 0 \ (mod \ 3)$ ise $x \equiv 0 (mod \ 3)$, ve $ry - sx = \pm 1$

3) $r, s \not\equiv 0 \ (mod \ 3)$ ise $x \not\equiv 0 (mod \ 3)$, $y \not\equiv 0 (mod \ 3)$ ve $ry - sx = \pm 1$
\[
\frac{1}{1} \rightarrow \frac{a_1}{b_1} \rightarrow \ldots \rightarrow \frac{a_k}{b_k} \rightarrow \frac{0}{1} \rightarrow \frac{1}{0},
\]
where some arrows may be reversed. From the above edge conditions, we easily see that \( \frac{1}{1} \leftrightarrow \frac{a_1}{b_1} \) if and only if \( a_1, b_1 \not\equiv 0(\text{mod}3) \). Then \( \frac{a_1}{b_1} \leftrightarrow \frac{a_2}{b_2} \) if and only if \( a_2, b_2 \not\equiv 0(\text{mod}3) \). If we proceed in this way, we obtain that \( \frac{a_k}{b_k} \leftrightarrow \frac{a_{k+1}}{b_{k+1}} \) if and only if \( a_{k+1}, b_{k+1} \not\equiv 0(\text{mod}3) \). This contradicts to \( \frac{a_{k+1}}{b_{k+1}} \leftrightarrow \frac{0}{1} \) since we get \( 0 \not\equiv 0(\text{mod}3) \) from the edge conditions. This shows that there is no path of the form \( D \). Similarly, we can show that there is no path of the form
\[
\frac{2}{1} \rightarrow \frac{x_1}{y_1} \rightarrow \ldots \rightarrow \frac{x_k}{y_k} \rightarrow \frac{x_{k+1}}{y_{k+1}} \rightarrow \frac{3}{1}.
\]
Consequently, \( F^3_{0,1} \) is not connected.
The subgraphs $F_{1,2}^3$, $F_{3,2}^3$, and $F_{5,2}^3$ are not connected.

All subgraphs $F_{u,2}^3$ are not connected.
The subgraph $F_{1,3}^3$, $F_{2,3}^3$, $F_{4,3}^3$, $F_{5,3}^3$, $F_{7,3}^3$ and $F_{8,3}^3$ are not connected.

All subgraphs $F_{u,3}^3$ are not connected.
The subgraph $F_{1,4}^3$, $F_{3,4}^3$, $F_{5,4}^3$, $F_{7,4}^3$, $F_{9,3}^3$ and $F_{11,3}^3$ are not connected.

All subgraphs $F_{u,4}^3$ are not connected.
Theorem

If $n \geq 5$, then $F^3_{u,n}$ is not connected.


THANKS