On restricted distributivity of aggregation operators and utility functions

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- **Restricted distributivity**
- **Utility function**
  - probabilistic mixture
  - possibilistic mixture
  - hybrid probabilistic-possibilistic mixture
  - hybrid probabilistic-possibilistic mixture extension

Preliminary notions

An aggregation operator is a function $A^{(n)} : [0,1]^n \longrightarrow [0,1]$ that is nondecreasing in each variable and that fulfills the following boundary conditions

$$A^{(n)}(0,\ldots,0) = 0 \quad \text{and} \quad A^{(n)}(1,\ldots,1) = 1.$$

A nullnorm F is a binary aggregation operator on $[0,1]$ that is commutative, associative and for which there exists an element k in $[0,1]$ such that

$$F(x,0) = x \text{ for } x \leq k \quad \text{and} \quad F(x,1) = x \text{ for } x \geq k.$$

k - absorbing element

T. Calvo, B. De Baets, J. Fodor, The functional equations of Frank and Alsina for uninorms and nullnorms, FSS 120 (2001) 385-394.

Preliminary notions

Let F be a nullnorm such that $k \in (0,1)$. Then,

$$F(x,y) = \begin{cases} kS\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)T\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{otherwise,} \end{cases}$$

where T is a t-norm, and S is a t-conorm.

M. Mas, G. Mayor, J. Torrens, t-operators, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems 7 (1999) 31-50.

Restricted distributivity

Let F, G be binary aggregation operators. F is distributive over G, if

(DL) F is a left distributive over G ,i.e.,

$$F(x, G(y, z)) = G(F(x, y), F(x, z)), \quad \text{for all } x, y, z \in [0, 1]$$

and

(DR) F is a right distributive over G ,i.e.,

$$F(G(y, z), x) = G(F(y, x), F(z, x)), \quad \text{for all } x, y, z \in [0, 1]$$

$G(y, z) < 1$ - **restricted distributivity**

Restricted distributivity - t-norm and t-conorm

A continuous t-norm T and a continuous t-conorm S satisfy **(RD)**, if and only if exactly one of the following cases is fulfilled:

(i) $S = S_M$

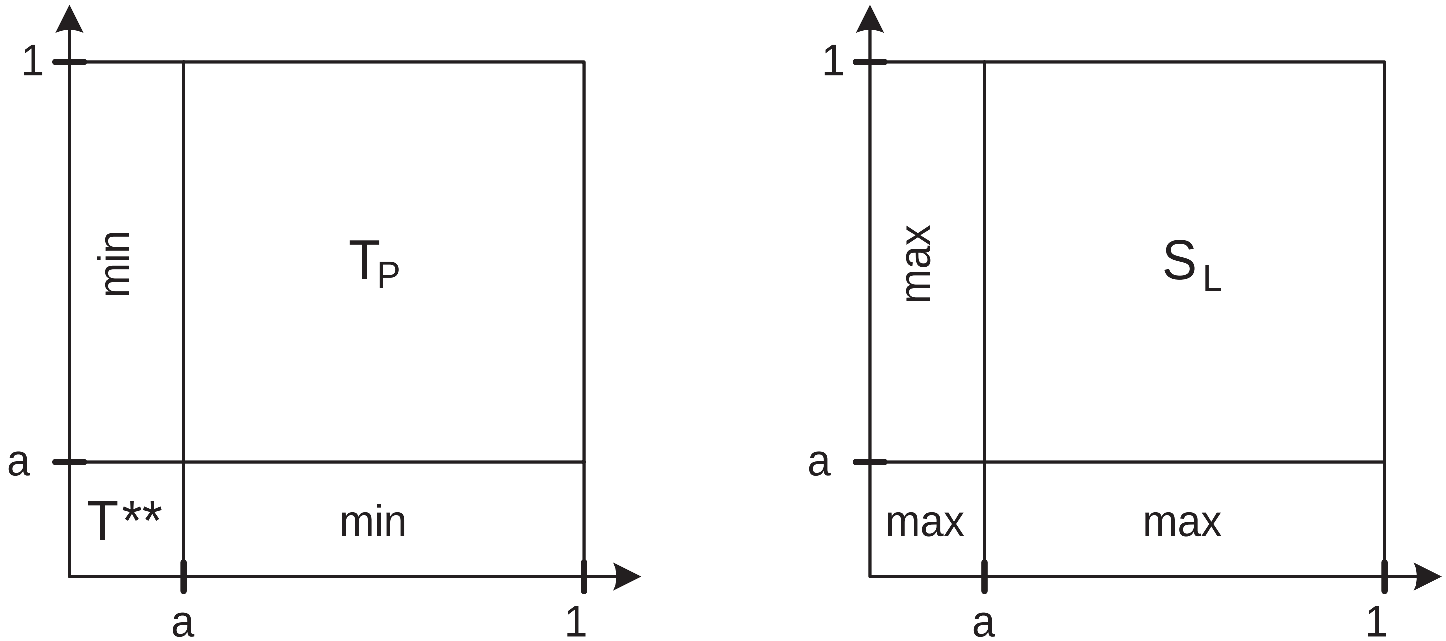
(ii) there is a strict t-norm T^* and a nilpotent t-conorm S^* such that additive generator s of S^* satisfying $s(1) = 1$ is also a multiplicative generator of T^*, and there is an $a \in [0, 1)$ such that for some continuous t-norm T^{**} the following holds:

$$S(x,y) = \begin{cases} a + (1-a)S^*\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{1}$$

and

$$T(x,y) = \begin{cases} aT^{**}\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)T^*\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{2}$$

Restricted distributivity - t-norm and t-conorm



Restricted distributivity: t-norm and t-conorm.

E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers , Dordrecht, 2000.

Restricted distributivity - nullnorm and t-conorm

A continuous nullnorm F with absorbing element $k \in (0,1)$ and a continuous t-conorm S satisfy (RD) if and only if exactly one of the following cases is fulfilled:

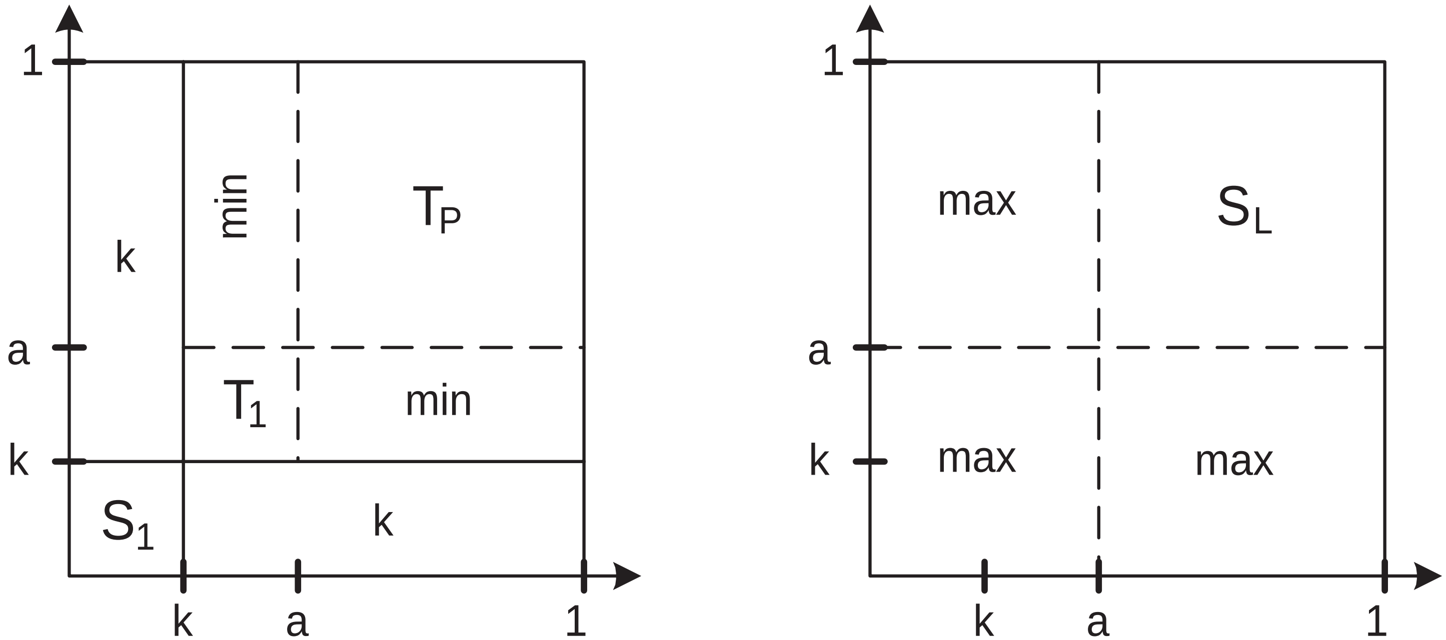
(i) $S = S_M$

(ii) there is an $a \in [k, 1)$ such that S is of the previous form and F is given by:

$$F(x,y) = \begin{cases} kS_1\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (a-k)T_1\left(\frac{x-k}{a-k}, \frac{y-k}{a-k}\right) & \text{if } (x,y) \in [k,a]^2, \\ a + (1-a)T_P\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ \min(x,y) & \text{if } k \le \min(x,y) \\ & \le a \le \max(x,y), \\ k & \text{otherwise,} \end{cases} \tag{3}$$

where S_1 is an arbitrary continuous t-conorm and T_1 is an arbitrary continuous t-norm.

Restricted distributivity - nullnorm and t-conorm



Restricted distributivity: nullnorm and t-conorm.

D. Jočić, I. Štajner-Papuga, Restricted distributivity for aggregation operators with absorbing element, FSS 224 (2013), 23-35.

Restricted distributivity - nullnorm and uninorm

A uninorm U is a binary operator $U : [0,1] \times [0,1] \to [0,1]$ that is commutative, associative, non-decreasing in each variable and for which there exists a neutral element $e \in [0,1]$, i.e., $U(x,e) = x$ for all $x \in [0,1]$.

Definitions of triangular norms and of triangular conorms are special cases of the previous definition, i.e., for $e = 1$ operator U is a t-norm denoted by T and for $e = 0$ operator U is a t-conorm denoted by S

Let U be a uninorm with a neutral element $e \in (0,1)$ such that both functions $U(x,1)$ and $U(x,0)$ are continuous except at the point $x = e$.

(i) If $U(0,1) = 0$, then

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{4}$$

where T is a t-norm, and S is a t-conorm.

(ii) If $U(0,1) = 1$, then

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{5}$$

where T is a t-norm, and S is a t-conorm.

J. C. Fodor, R. R. Yager, A. Rybalov, Structure of uninorms, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems 5 (1997), 411-427.

Restricted distributivity - nullnorm and uninorm

A continuous nullnorm F with an absorbing element $k \in (0,1)$ and a uninorm $U \in U_{\max}$ with a continuous underlying t-norm and a continuous underlying t-conorm satisfy (RD) if and only if $e < k$ and exactly one of the following cases is fulfilled:

(i) F, U are given as in

M. Mas, G. Mayor, J. Torrens, The distributivity condition for uninorms and t-operators, Fuzzy Sets and Systems 128 (2002), 209-225.

(ii) There is $a \in [k, 1)$ such that F and U are given by

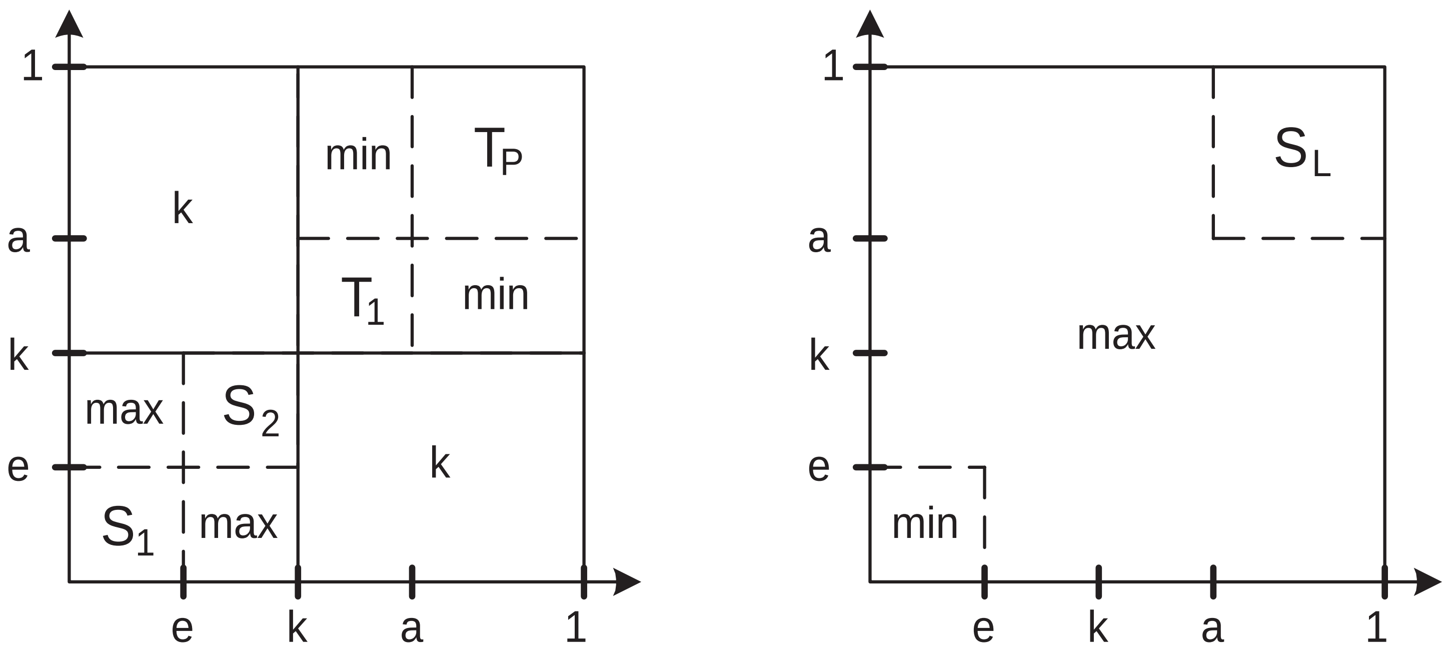
$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e]^2, \\ a + (1-a)S_L\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ \max(x,y) & \text{otherwise} \end{cases} \tag{6}$$

and

$$F(x,y)=\begin{cases} eS_1\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y)\in[0,e]^2,\\ e+(k-e)S_2\left(\frac{x-e}{k-e},\frac{y-e}{k-e}\right) & \text{if } (x,y)\in[e,k]^2,\\ k+(a-k)T_1\left(\frac{x-k}{a-k},\frac{y-k}{a-k}\right) & \text{if } (x,y)\in[k,a]^2,\\ a+(1-a)T_P\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) & \text{if } (x,y)\in[a,1]^2,\\ \max(x,y) & \text{if } \min(x,y)\le e\le\max(x,y)\le k,\\ \min(x,y) & \text{if } k\le\min(x,y)\le a\le\max(x,y),\\ k & \text{otherwise,}\end{cases} \tag{7}$$

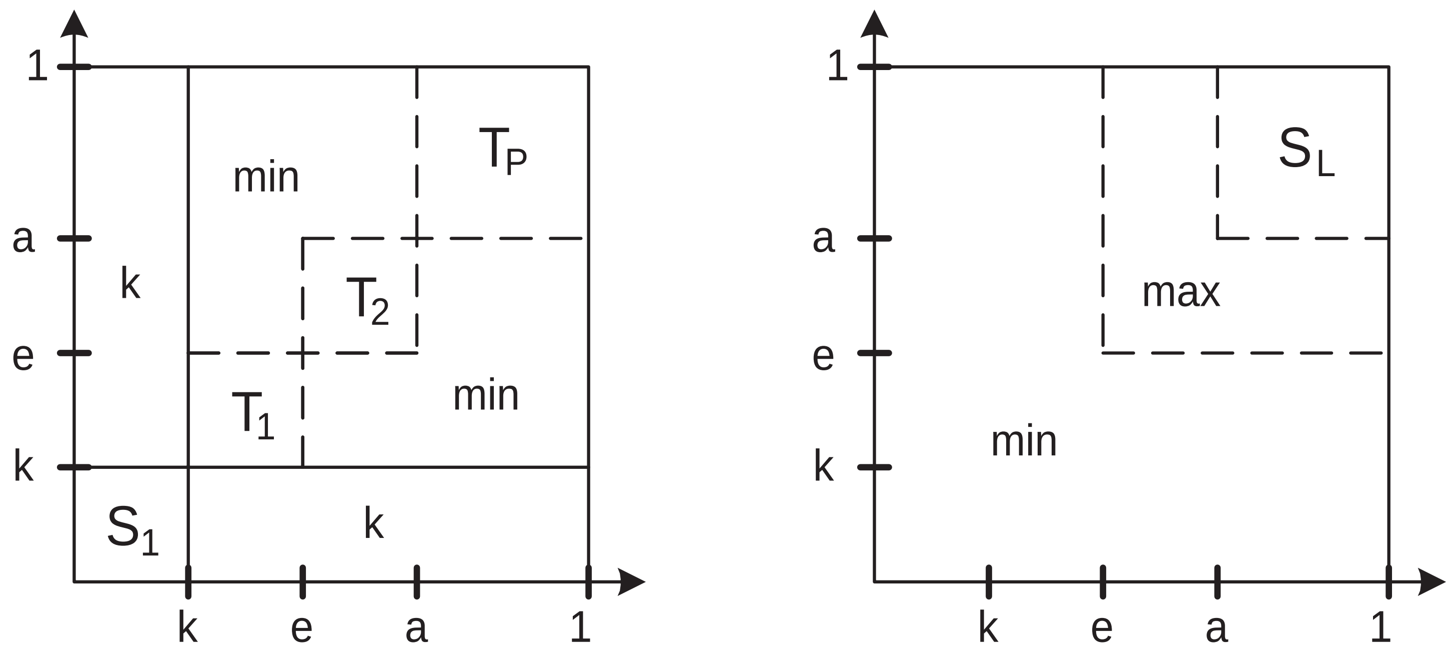
where S_1 and S_2 are continuous t-conorms and T_1 is a continuous t-norm.

Restricted distributivity - nullnorm and uninorm



Restricted distributivity: nullnorm and uninorm from $U_{\max}$.

Restricted distributivity - nullnorm and uninorm



Restricted distributivity: nullnorm and uninorm from $U_{\min}$.

Utility function

- probabilistic mixture
- possibilistic mixture
- hybrid probabilistic-possibilistic mixture

Utility function - probabilistic mixture

Combines two probability distributions into a new one:

$$V(p, p'; \alpha, \beta) = \alpha \cdot p + \beta \cdot p'$$

$\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$

J. von Neumman, O. Morgenstern, Theory of Games and Economic Behavior, Princeton Uviversity Press, Princeton 1944.

Utility function - possibilistic mixture

Combines two possibility distributions into a new one:

$$V(\pi, \pi'; \alpha, \beta) = \max(\min(\alpha, \pi), \min(\beta, \pi'))$$

α, β are from a valuation scale and $\max(\alpha, \beta) = 1$

D. Dubois, L. Godo, H. Prade, A. Zapico, Making decision in qualitative setting: from decision under uncertainty to case-based decision, (Eds. A.G. Cohn, L. Schurbet, S.C. Shapiro) Proceedings of the 6th Interational Conference, Principles of Knowledge Representation and Reasoning, San Francisko (1998), 594-605.

Utility function - hybrid mixture

Combines two S-measures into a new one:

$$M(m, m'; \alpha, \beta) = S(T(\alpha, m), T(\beta, m'))$$

where (S, T) is a pair of continuous t-conorm and t-norm satisfying (RD) and pair (α, β) is from $\Phi_{S,a}$ given by

$$\Phi_{S,a} = \{(\alpha, \beta) | \alpha, \beta \in (0, 1), \alpha + \beta = 1 + a \text{ or } \min(\alpha, \beta) \leq a, \max(\alpha, \beta) = 1\}$$

Utility function - hybrid mixture

The optimistic hybrid utility function by means of hybrid mixtures

$$U(u_1, u_2; \mu_1, \mu_2) = S(T(u_1, \mu_1), T(u_2, \mu_2))$$

u_1, u_2 - utilities with values in $[0, 1]$; μ_1, μ_2 - degrees of plausibility from $\Phi_{S,a}$.

The pessimistic hybrid utility function $\overline{U}$:

$$\overline{U}(u_1, u_2; \mu_1, \mu_2) = 1 - U(1 - u_1, 1 - u_2; \mu_1, \mu_2).$$

D. Dubois, J. Fodor, H. Prade, M. Roubens, Aggregation of decomposable measures with applications to utility theory, Theory and Decision 41 (1996), 59-95.
D. Dubois, E. Pap, H. Prade, Hybrid probabilictic-possibilistic mixtures and utility functions, Preferences and Decisions under Incomplete Knowledge, Studies i Fuzziness and Soft Computing, vol. 51, 2000, 51-73.

Utility function - nullnorm extension

(S, F) is a pair of continuous t-conorm and continuous nullnorm satisfying (RD) and pair (α, β) is from $\Phi_{S,a}$

$$U_F(u_1, u_2; \mu_1, \mu_2) = S(F(u_1, \mu_1), F(u_2, \mu_2))$$

Utility function - nullnorm extension - behavioral characteristics

Case I: $\mu_1 > a, \mu_2 > a$

1. Let $u_1 > a,\ u_2 > a$. Then

$$U_F(u_1, u_2; \mu_1, \mu_2) = \frac{u_1(\mu_1 - a) + u_2(1 - \mu_1)}{1 - a}.$$

2. Let $u_2 > a \geq u_1$. Then

$$U_F(u_1, u_2; \mu_1, \mu_2) = a + \frac{(u_2 - a)(\mu_2 - a)}{1 - a}.$$

3. Let $u_1 \leq a,\ u_2 \leq a$. Then:

   (a) if $u_1 \leq k,\ u_2 \leq k$, then $U_F(u_1, u_2; \mu_1, \mu_2) = S(k, k) = \max(k, k) = k$;

   (b) if $u_2 \leq k < u_1$, then $U_F(u_1, u_2; \mu_1, \mu_2) = S(u_1, k) = \max(u_1, k) = u_1$;

   (c) if $u_1 \leq k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = S(k, u_2) = \max(u_2, k) = u_2$;

   (d) if $k < u_1,\ k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = S(u_1, u_2) = \max(u_1, u_2)$.

Utility function - nullnorm extension - behavioral characteristics

Case II a): $\mu_1 \leq k, \mu_2 = 1$

1. If $u_1 > a,\ u_2 > a$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, u_2) = u_2$.
2. If $u_1 \leq a,\ u_2 \leq a$, then:
   (a) if $u_1 \leq k,\ u_2 \leq k$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(S_1(u_1, \mu_1), k) = k$;
   (b) if $u_2 \leq k < u_1$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, k) = k$;
   (c) if $u_1 \leq k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(S_1(u_1, \mu_1), u_2) = u_2$;
   (d) if $k < u_1,\ k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, u_2) = u_2$.
3. If $u_1 \leq a < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = u_2$.
4. If $u_2 \leq a < u_1$, then:
   (a) if $u_2 \leq k$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, k) = k$;
   (b) if $u_2 > k$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, u_2) = u_2$.

Utility function - nullnorm extension - behavioral characteristics

Case II b): $\mu_1 \in (k, a], \mu_2 = 1$

1. If $u_1 > a,\ u_2 > a$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(\mu_1, u_2) = u_2$.
2. If $u_1 \leq a,\ u_2 \leq a$, then:
   (a) if $u_1 \leq k,\ u_2 \leq k$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, k) = k$;
   (b) if $u_1 \leq k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(k, u_2) = u_2$;
   (c) if $u_2 \leq k < u_1$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(T_1(u_1, \mu_1), k) = T_1(u_1, \mu_1)$;
   (d) if $k < u_1,\ k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(T_1(u_1, \mu_1), u_2)$.
3. If $u_1 \leq a < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = u_2$.
4. If $u_2 \leq a < u_1$, then:
   (a) if $u_2 \leq k$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(\mu_1, k) = \mu_1$;
   (b) if $k < u_2$, then $U_F(u_1, u_2; \mu_1, \mu_2) = \max(\mu_1, u_2)$.