Generalized Uniform Fuzzy Partition

Analysis of Necessary and Sufficient Condition

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EVROPSKÁ UNIE
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OP Výzkum a vývoj pro inovace

Partitioning of the universe using fuzzy sets

- Ruspini (1969) was probably the first who relaxed the crisp borders of equivalence classes using fuzzy sets in his new approach to clustering. His idea initiated a deep research in the field of fuzzy partitions.
- Principally, we can recognized two basic directions in the definition of fuzzy partitions:
  - a generalization of the covering and disjointness axioms of standard partition. This direction was followed by e.g. Butnariu (1983), Ovchinnikov (1991), Dumitrescu (1992) and Klement, Moser (1997).
  - searching for appropriate axioms of fuzzy partitions ensuring the one-to-one correspondence between fuzzy partitions and fuzzy similarity relations. This direction was followed by e.g. Bhakat, Das (1992), Höhle (1996), Mesiar, Reusch, Thiele (1998) and Montes, Couso, Gil (2001).

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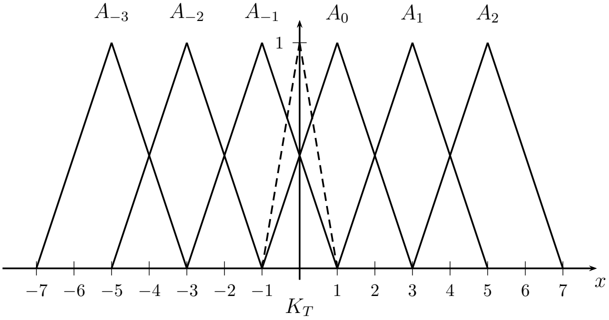
Application of fuzzy partitions

The use of fuzzy partitions can be recognized in various fields of application of fuzzy set theory:

1. fuzzy pattern recognition - Ruspini (1969), Bezdek (1978), Yang (1993)
2. fuzzy control - Höhle (1998), Moser (2009)
3. fuzzy relations equations - Perfilieva, Novák (2007), Štěpnička, de Baets, Nosková (2010)
4. fuzzy histogram estimation - Waltman, Kaymak, van den Berg (2005), Loquin, Strauss (2008)
5. fuzzy transform - Perfilieva (2004)

Uniform fuzzy partition (UFP) of triangular type



two overlapping functions (K, h, x_0), $x_k = x_0 + kh$

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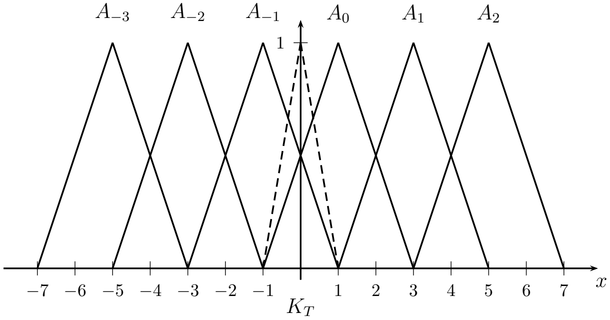

$$K_T(x) = \max(1 - |x|, 0),\ h = 2,\ x_0 = 1$$

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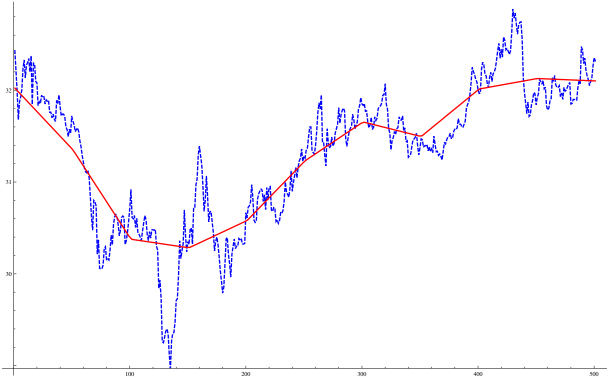
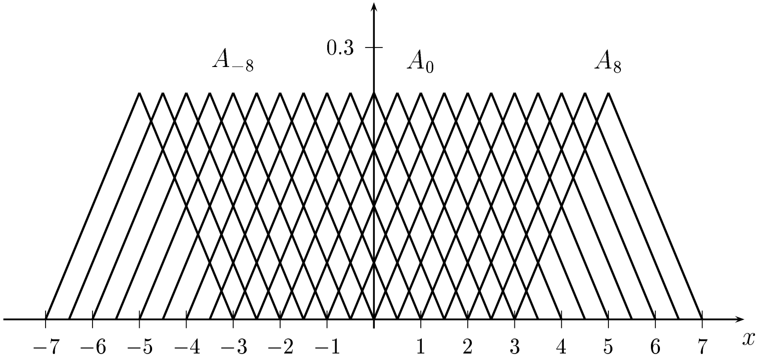
Trend estimation using F-transform based on UFP



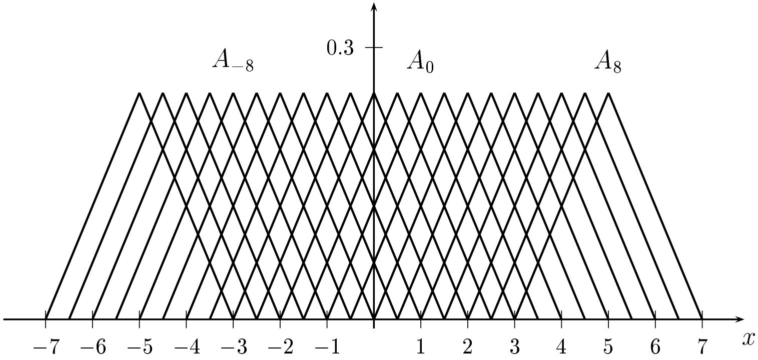
$$\hat{f}(t) = \sum_{k=1}^{\ell} \frac{\sum_{i=1}^{N} f(t_i) K_T\left(\frac{t_i - x_k}{h}\right)}{\sum_{i=1}^{N} K_T\left(\frac{t_i - x_k}{h}\right)} K_T\left(\frac{t - x_k}{h}\right)$$

Generalized uniform fuzzy partition (GUFP) of triangular type



more than two overlapping functions (K, h, r, x_0), $x_k = x_0 + kr$

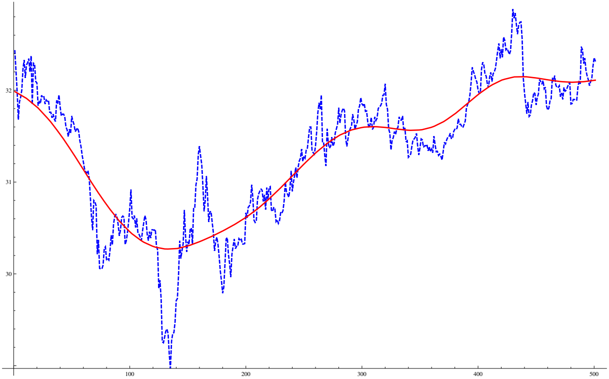
Generalized uniform fuzzy partition (GUFP) of triangular type



$$K_T(x) = \max(1 - |x|, 0),\ h = 2, r = 0.5,\ x_0 = 1$$

more than two overlapping functions (K, h, r, x_0), $x_k = x_0 + kr$

Trend estimation using F-transform based on GUFP



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Generating function

Definition

A function $K : \mathbb{R} \to [0,1]$ *is said to be a generating function if* K *is an even integrable function (fuzzy set) which is non-increasing in* $[0,\infty)$ *and*

$$K(x) \begin{cases} > 0, & \text{if } x \in (-1,1); \\ = 0, & \text{otherwise.} \end{cases}$$

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Remark

In contrast to the standard approach to UFP, we use generating functions which are not normal in general.

Generalized uniform fuzzy partition

Definition

Let K be a normal generating function, h and r be positive real numbers and $x_0 \in \mathbb{R}$. A system of fuzzy sets $\{A_i \mid i \in \mathbb{Z}\}$ defined by

$$A_i(x) = K\left(\frac{x - x_0 - i\,r}{h}\right)$$

for any $i \in \mathbb{Z}$ is said to be a generalized uniform fuzzy partition (GUFP) of the real line determined by the quadruplet (K, h, r, x_0) if the Ruspini's condition is satisfied, i.e.,

$$S(x) = \sum_{i \in \mathbb{Z}} A_i(x) = 1$$

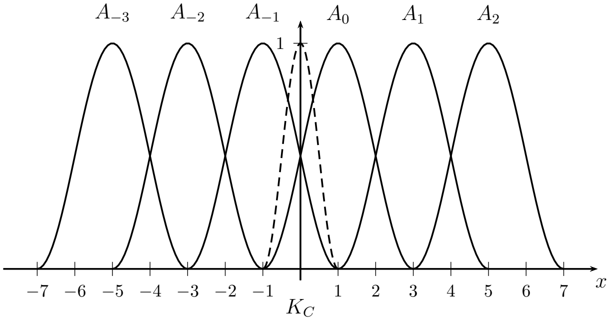
holds for any $x \in \mathbb{R}$. The parameters h, r and x_0 are called spread, shift and central node, respectively.

Uniform fuzzy partition

Definition

A generalized uniform fuzzy partition (K, h, r, x_0) is said to be a uniform fuzzy partition (UFP) if $h = r$.

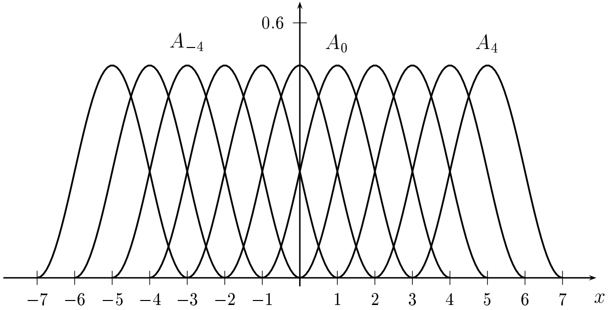
Raised cosine uniform fuzzy partition



A part of the GUFP of the real line determined by $(K_T, 2, 2, 1)$.

$\boxed{\text{h=r}}$

Raised cosine generalized uniform fuzzy partition



A part of the GUFP of the real line determined by $(K_C, 2, 1, 0)$,

$\boxed{\text{h=2r}}$

Natural questions

Q1: Can we provide a "simple condition" concerning K, h, and r under which (K, h, r, x_0) determines a uniform fuzzy partition of the real line?

Q2: Are this condition necessary and sufficient, it means, equivalent to Ruspini's one?

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Q1: Can we provide a "simple condition" concerning K, h, and r under which (K, h, r, x_0) determines a uniform fuzzy partition of the real line?

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Simple observation for type (K, h, h, x_0) GUFP

Type (K_T, h, h, x_0)

Let K_T be the triangular generating function and $y \in [\frac{1}{2}, 1]$. Then,

$$\int_{1-y}^{y} K_T(x)dx = \int_{1-y}^{y} (1-x)dx = \left[x - \frac{x^2}{2}\right]_{1-y}^{y} = y - \frac{1}{2}.$$

Simple observation for type (K, h, h, x_0) GUFP

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Type (K_C, h, h, x_0)

Let K_C be the raised cosine generating function and $y \in [\frac{1}{2}, 1]$. Then,

$$\int_{1-y}^{y} K_C(x)dx = \int_{1-y}^{y} \frac{1}{2}(1 + \cos(\pi x))dx = \left[\frac{x}{2} + \frac{\sin(\pi x)}{2\pi}\right]_{1-y}^{y} = y - \frac{1}{2}.$$

Main theorem for UFP - (K, h, h, x_0) type

Theorem

A quadruplet (K, h, h, x_0) *determines a GUFP iff*

$$\int_{1-y}^{y} K(x)dx = y - \frac{1}{2}$$

holds for any $y \in [\frac{1}{2}, 1]$.

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holds for any $y \in [\frac{1}{2}, 1]$.

Corollary

If a quadruplet (K, h, h, x_0) *determines a GUFP, then*

$$\int_{-\infty}^{\infty} K(x)dx = 1.$$

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Important lemma for type (K, h, r, x_0) - a generalization of the previous corollary

Lemma

If a quadruplet (K, h, r, x_0) *determines a GUFP, then*

$$\int_{-\infty}^{\infty} K(x)dx = \frac{r}{h}.$$

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Main theorem for type (K, h, r, x_0) GUFP

Theorem

Put $\alpha = \frac{r}{h}$. *Then, a quadruplet* (K, h, r, x_0) *determines a generalized uniform fuzzy partition iff*

$$\sum_{i=1}^{\infty} \int_{i\alpha - y}^{y+(i-1)\alpha} K(x)dx = y - \frac{\alpha}{2}$$

holds for any $y \in [\frac{\alpha}{2}, \alpha]$.

Alternative expression of main theorem for type (K, h, r, x_0) GUFP

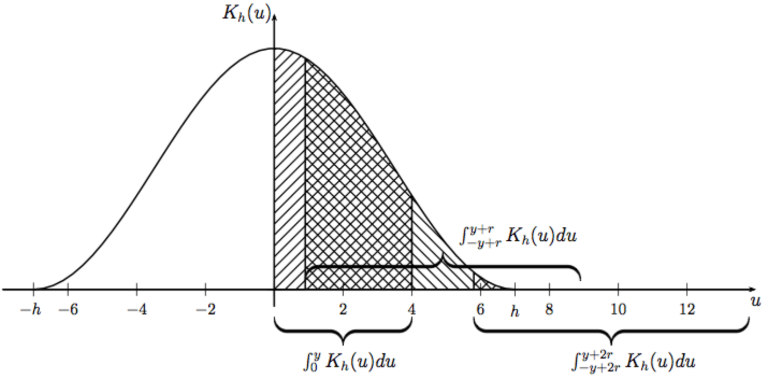
Theorem

Put $K_h(x) = K(\frac{x}{h})$. *A quadruplet* (K, h, r, x_0) *determines a generalized uniform fuzzy partition iff*

$$\sum_{i=1}^{\infty} \int_{ir-y}^{y+(i-1)r} K_h(x)dx = y - \frac{r}{2}$$

holds for any $y \in [\frac{r}{2}, r]$.

Graphical illustration of the previous equality



The content of crosshatched surfaces is equal to $y - \frac{r}{2}$.
$(y = 4,\ h = 7,\ r = 4.9)$

Remark

It is easy to see that (K, h, h, x_0) *(i.e.,* $\alpha = \frac{r}{h} = 1$*) determines a uniform fuzzy partition iff*

$$y - \frac{1}{2} = \sum_{i=1}^{\infty} \int_{i-y}^{y+i-1} K(x)dx = \int_{1-y}^{y} K(x)dx,$$

for any $y \in [\frac{1}{2}, 1]$.

A straightforward consequence of the main theorem

Corollary

If (K, h, r, x_0) *determines a generalized uniform fuzzy partition and* $\beta > 0$ *is a real number then* $(K, \beta h, \beta r, x_0)$ *determines it as well.*

A straightforward consequence of the main theorem

Corollary

If (K, h, r, x_0) determines a generalized uniform fuzzy partition and $\beta > 0$ is a real number then $(K, \beta h, \beta r, x_0)$ determines it as well.

Proof.

According to the previous theorem, it follows from $\frac{r}{h} = \frac{\beta r}{\beta h}$. □

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A necessary and sufficient condition for particular cases

Denotation

If K is a normal generating function, we use $\mathrm{GF}(K)$ to denote the family of all function $\alpha \odot K$ for $\alpha \in (0, 1]$, where $\alpha \odot K(x) = \alpha K(x)$.

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Theorem

Let $K \in \mathrm{GF}(K_T)$ *or* $K \in \mathrm{GF}(K_C)$ *such that* $\int_{-1}^{1} K(x)dx = \frac{r}{h}$. *Then,* (K, h, r, x_0) *determines a GUFP iff* $\frac{h}{r} \in \mathbb{N}$.

Open question

An open question is whether the previous two results may be generalized for an arbitrary normal generating function K (e.g., defined by splines) with or without the following assumption:

$$\int_{-1}^{1} K(x)dx = 1.$$

Hypotheses

Let K be a normal generating function (satisfying or not the previous integral equality) and $L \in \mathrm{GF}(K)$ such that $\int_{-1}^{1} L(x)dx = \frac{h}{r}$. Then, (L, h, r, x_0) determines a GUFP iff $\frac{h}{r} \in \mathbb{N}$.

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Conlcusion

- We defined a generalized uniform fuzzy partitions (GUFP) as a quadruplet (K, h, r, x_0).
- We found an integral necessary and sufficient condition for GUFP equivalent to Ruspini's condition.
- We simplified this condition for the triangular and raised cosine type of GUFP.
- We provided an open question concerning a generalization of the previous simplification.

Thank you for your attention.

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