(Pre-)orders induced by uninorms
Our motivation for studying of uninorms

Problem
What is the behavior of non-representable uninorms, particularly, conjunctive uninorms in the rectangle $[0, e] \times [e, 1]$?

Why is it interesting? Part 1
This problem occurs, e.g., when constructing fuzzy implications of the form

$$I(x, y) = n_1(U(x, n_2(y))),$$

where $n_1$ and $n_2$ are suitably chosen negations, and studying especially the neutrality property ($I(x, 0) = x$) and the exchange principle ($I(x, I(y, z)) = I(y, I(x, z))$).
Why is it interesting? Part 2

The behaviour of uninorms in the rectangle $]0, e[ \times ]e, 1]$ is important also when studying the (pre-)order generated from uninorms.
Why is it interesting? Part 2

The behaviour of uninorms in the rectangle $[0, e] \times [e, 1]$ is important also when studying the (pre-)order generated from uninorms.


Definition

Let $L$ be a bounded lattice, $T$ be a t-norm on $L$. Then the order

$$x \preceq_T y \iff (\exists \ell \in L) T(\ell, y) = x$$

is called a t-order for the t-norm $T$.

$$x \preceq_T y \implies x \leq y$$
A uninorm $U$ is a function $U : [0, 1]^2 \to [0, 1]$ that is increasing, commutative, associative and has a neutral element $e \in [0, 1]$. A uninorm $U$ is said to be conjunctive if $U(x, 0) = 0$, and $U$ is said to be disjunctive if $U(1, x) = 1$, for all $x \in [0, 1]$. 

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Rewriting the definition of t-order we get:

Let $U$ be a uninorm. We introduce the following relation

$$x \preceq_U y \iff (\exists \ell \in [0, 1]) U(\ell, y) = x.$$
For an arbitrary uninorm $U$ and arbitrary $(x, y) \in ]0, e[ \times ]e, 1[ \cup ]e, 1[ \times ]0, e[$ we have
\[
\min\{x, y\} \leq U(x, y) \leq \max\{x, y\}.
\]

Let $U$ be a uninorm with $e \in ]0, 1[$. Then we denote
\[
T_U = U \upharpoonright [0, e]^2, \quad S_U = U \upharpoonright [e, 1]^2.
\]

$\mathcal{U}\{x,y\}$ denotes those uninorms attaining at each point of $]0, e[ \times ]e, 1[ \cup ]e, 1[ \times ]0, e[$ the lower or the upper bound.
Lemma

Let $U \in \mathcal{U}_{\{x, y\}}$. Then:

1. $\preceq_U$ is an order,
2. $(\forall x, y \in [0, e]) (x \preceq_U y \iff x \preceq_{T_U} y),$
3. $(\forall x, y \in [e, 1]) (x \preceq_U y \iff x \preceq_{S_U} y).$
Using transformation of $T_M$ in the square $[0, e]^2$ and $S_M$ in the square $[e, 1]^2$

\[ U_1(x, y) = \begin{cases} 
\max\{x, y\}, & \text{if } \min\{x, y\} \geq \frac{1}{2}, \\
\min\{x, y\}, & \text{if } \frac{1}{4} < \min\{x, y\} < \frac{1}{2}, \text{ or if } \max\{x, y\} = \frac{1}{2}, \\
\frac{1}{4}, & \text{if } 0 < \min\{x, y\} \leq \frac{1}{4} \text{ and } \max\{x, y\} > \frac{1}{2}, \\
0, & \text{otherwise.}
\end{cases} \]
Using transformation of $T_M$ in the square $[0, e]^2$ and $S_M$ in the square $[e, 1]^2$

$$U_2(x, y) = \begin{cases} 
\max\{x, y\}, & \text{if } \min\{x, y\} \geq \frac{1}{2}, \\
\min\{x, y\}, & \text{if } \max\{x, y\} = e, \\
0, & \text{if } \min\{x, y\} \leq \frac{1}{4} \text{ and } \max\{x, y\} < \frac{1}{2}, \\
\frac{2^i - 1}{2^{i+1}}, & \text{for } i \in \{1, 2, 3, \ldots \}, \\
\text{if } \frac{2^i - 1 - 1}{2^i} < \min\{x, y\} \leq \frac{2^i - 1}{2^{i+1}} \text{ and } \max\{x, y\} > \frac{1}{2}, \\
\text{or if } \frac{2^i - 1}{2^{i+1}} < \min\{x, y\} \leq \frac{2^i + 1 - 1}{2^{i+2}} \text{ and } \max\{x, y\} < \frac{1}{2}.
\end{cases}$$
Let $U_r$ be a representable uninorm.

$U_3(x, y) = \begin{cases} 
\frac{1}{2} U_r(2(x - \frac{1}{4}), 2(y - \frac{1}{4})) + \frac{1}{4}, & \text{if } (x, y) \in [\frac{1}{4}, \frac{3}{4}]^2, \\
\max\{x, y\}, & \text{if } \max\{x, y\} > \frac{3}{4} \text{ and } \min\{x, y\} > 0.1, \\
0, & \text{if } \max\{x, y\} \leq 0.1 \text{ or if } \min\{x, y\} = 0, \\
0.1, & \text{if } \max\{x, y\} > \frac{3}{4} \text{ and } 0 < \min\{x, y\} \leq 0.1, \\
\min\{x, y\}, & \text{otherwise}. 
\end{cases}$

In the uninorm $U_3$ we could change $S_M$ for an arbitrary t-conorm on the square $[\frac{3}{4}, 1]^2$ without changing other values.
\[
\frac{1}{2} U_r(2(x - \frac{1}{4}), 2(y - \frac{1}{4})) + \frac{1}{4}
\]
Using transformation of $T_D$ in the square $[0, e]^2$, and $S_M$ in the square $[e, 1]^2$

$$U_4(x, y) = \begin{cases} 
\max\{x, y\}, & \text{if } \min\{x, y\} \geq 0.5, \\
\min\{x, y\}, & \text{if } 0.5 \leq \max\{x, y\} \leq 0.8 \text{ and } \min\{x, y\} < 0.5, \\
0.1, & \text{if } 0.1 \leq \min\{x, y\} < 0.2 \text{ and } 0.3 < \max\{x, y\} < 0.5, \\
0.2, & \text{if } 0.2 \leq \min\{x, y\} < 0.3 \text{ and } 0.3 < \max\{x, y\} < 0.5, \\
0.3, & \text{if } 0.3 \leq x < 0.5 \text{ and } 0.3 \leq y < 0.5, \\
0, & \text{if } 0.2 \leq \min\{x, y\} \leq 0.3 \text{ and } \max\{x, y\} > 0.8, \\
0, & \text{otherwise,}
\end{cases}$$
Using transformation of $T_D$ in the square $[0, e]^2$, and $S_D$ in the square $[e, 1]^2$

\[
U_5(x, y) = \begin{cases} 
1, & \text{if } \min\{x, y\} > 0.5, \text{ or if } \max\{x, y\} = 1 \text{ and } \min\{x, y\} > 0.2, \\
\max\{x, y\}, & \text{if } \max\{x, y\} > 0.5 \text{ and } 0.4 < \min\{x, y\} \leq 0.5, \\
\frac{3}{4}, & \text{or if } \frac{9}{16} \leq \max\{x, y\} < \frac{3}{4} \text{ and } 0.2 < \min\{x, y\} \leq 0.4, \\
\frac{2^i+1}{2^{i+1}}, & \text{if } \frac{2^i+1}{2^{i+1}} \leq \max\{x, y\} < \frac{2^{i-1}+1}{2^i} \text{ and } 0.2 < \min\{x, y\} \leq 0.4, \\
0.2, & \text{for } i \in \{4, 5, 6, \ldots\}, \\
\min\{x, y\}, & \text{if } \max\{x, y\} > 0.5 \text{ and } 0 < \min\{x, y\} \leq 0.2, \\
0, & \text{if } 0.2 < \max\{x, y\} \leq 0.5 \text{ and } 0 < \min\{x, y\} \leq 0.5, \\
\text{otherwise.} & 
\end{cases}
\]
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Proposition 1

If $U$ is a representable uninorm or if there exists a square $[a, b]^2$ with a representable uninorm, then the relation $\preceq_U$ is not an ordering.

Proposition 2

Let $U \notin U_{x,y}$. Then at least one of the following equivalences is violated:

- $(\forall x, y \in [0, e]) (x \preceq_U y \Leftrightarrow x \preceq_{T_U} y)$,
- $(\forall x, y \in [e, 1]) (x \preceq_U y \Leftrightarrow x \preceq_{S_U} y)$. 
Disjunctive uninorm

\[
\begin{array}{c}
\text{max} \\
\text{min} \\
0 \\
0.5 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
\left[\frac{1}{2}, 1\right] \\
\left[0, \frac{1}{2}\right] \\
0 \\
1 \\
\end{array}
\]
Conjunctive uninorm

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Conjunctive uninorm

\[
\begin{array}{|c|c|}
\hline
0 & \frac{1}{4} \\
\hline
\frac{1}{4} & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
0 & \frac{1}{4} \\
\hline
\frac{1}{4} & 1 \\
\hline
\end{array}
\]
(Pre-)orderings generated by uninorms $U_1 - U_3$
Conjunctive uninorm
Conjunctive uninorm

\[
\begin{array}{cccc}
1 & \frac{1}{4} & \frac{3}{8} & \frac{7}{16} & 0.5 \\
\frac{1}{4} & \frac{3}{8} & \frac{7}{16} & \cdots & \text{max} \\
0 & \frac{3}{8} & \frac{7}{16} & \cdots & \frac{7}{16} \\
0 & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[\frac{1}{2} \rightarrow \left[\frac{1}{2}, 1\right] \rightarrow \left[\frac{1}{3}, \frac{3}{8}\right] \rightarrow \frac{7}{16}\]

\[\left[0, \frac{1}{4}\right] \rightarrow \frac{1}{4} \rightarrow 0\]
Conjunctive uninorm, $U_r$ is conjunctive

\[ \frac{1}{2} U_r(2(x - \frac{1}{4}), 2(y - \frac{1}{4})) + \frac{1}{4} \]

\[ 0, 0.1 \]

\[ 0.1 \]

\[ \frac{1}{4}, \frac{3}{4} \]

\[ \frac{3}{4}, 1 \]

\[ 0, 0.1 \left[ \begin{array}{c} 0.1, \frac{1}{4} \end{array} \right] \]

\[ \left[ \begin{array}{c} \frac{1}{4}, \frac{3}{4} \end{array} \right] \]

\[ \text{ekv.} \]