# Cogenerators in generalized probability

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# STARRING: COGENERATOR

COGENERATOR

$=$ a set carrying a structure suitable

- to evaluate random events
- to capture logical operations on the events

- inside powers of the cogenerator live domains of generalized probability (models of random experiments)
- serves as the range of a generalized probability measure

**Today's programme:**

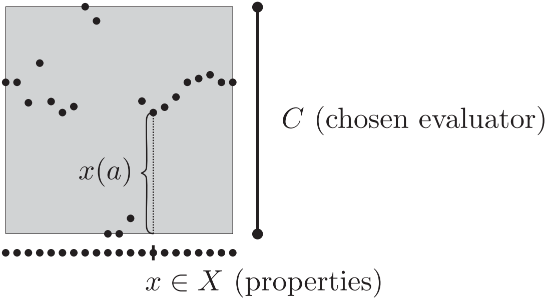
two traditional cogenerators $\{0,1\}$ and $[0,1]$, and their relationships $+$ properties of the corresponding probability domains

GOOD LANGUAGE for this job: a categorical approach

- Start with a "system $\mathcal{A}$ of events";
- Choose an "evaluator $C$"—a cogenerator; usually a structured set suitable for "evaluating" (e.g. the two element Boolean algebra, unit interval carrying Łukasiewicz MV-structure, D-poset, simplex, ...);
- Choose a set $X$ of "properties" evaluated via $C$ such that $X$ separates $\mathcal{A}$;
- Represent each event $a \in \mathcal{A}$ via the "evaluation" of $\mathcal{A}$ into $C^X$ assigning each $a \in \mathcal{A}$ its evaluation $a_X \in C^X$, $a_X \equiv \{x(a);\ x \in X\}$;
- Form the minimal "subalgebra" $D$ of $C^X$ containing $\{a_X;\ a \in \mathcal{A}\}$;
- The subalgebra forms a *probability domain* $D \subseteq C^X$.

# Representation of an event

| | |
|---|---|
| piece of reality | reality captured by evaluator |
| $a$ (event) $\longmapsto$ | $a_X$ (evaluation of $a$) |



# Goal

**Domain of classical probability theory** (CPT):

- random events form a $\sigma$-field $\mathbb{A}$ of sets
- cogenerated by the two-element Boolean algebra $\{0,1\}$

**Domain of fuzzy probability theory** (FPT):

- random events form an MV-algebra of $[0,1]$-valued measurable functions $\mathcal{M}(\mathbb{A})$
- cogenerated by the MV-algebra $[0,1]$

**OUR GOAL:** to characterize the transition from $\{0,1\}$ to $[0,1]$ and from $\mathbb{A}$ to $\mathcal{M}(\mathbb{A})$

# $\#1$ is the unit interval as D-poset

Based on our previous results in

R. Frič and M. Papčo.
On probability domains.
*Internat. J. Theoret. Phys.*, 49:3092–3100, 2010,

R. Frič and M. Papčo.
On probability domains II.
*Internat. J. Theoret. Phys.*, 50:3778–3786, 2011,

we claim that:

The unit interval $I = [0, 1]$ considered as a D-poset is the cogenerator which yields a suitable reference category, namely the category ID of D-posets of fuzzy sets and sequentially continuous D-homomorphisms.

# D-posets

- introduced by F. Kôpka & F. Chovanec in order to model quantum phenomena (effect algebras introduced by J. Foulis & M.K. Bennett are isomorphic structures)
- generalization of various structures, e.g. D-lattices, orthoalgebras, Boolean algebras, MV-algebras
- a category in which states and observables become morphisms
- **D-poset** is a partially order set $X$ with the least element $0_X$, the greatest element $1_X$, and a partial binary operation called **difference**, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for every $a \in X$;
(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$

- D-homomorphism: preserves the structure (order and difference)

# D-posets of fuzzy sets

Essential to probabilistic applications: **D-posets of fuzzy sets**
ID-poset = system $\mathcal{X} \subseteq I^X$ equipped with

- the coordinatewise partial order,
- coordinatewise convergence of sequences,
- the bottom and the top element of $I^X$,
- and closed with respect to the partial operation difference defined coordinatewise.

We always assume that $\mathcal{X}$ is REDUCED: for each $x, y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$.

D-posets of fuzzy sets as objects and sequentially continuous D-homomorphisms as morphisms = **the category ID**
(objects of ID are subobjects of the powers $I^X$)

# Morphisms

Let $(X, \mathbb{A}, p)$ be a classical probability space.
$\Rightarrow$ the probability integral $\tilde{p}(f) = \int f\,dp$ is a sequentially continuous D-homomorphism

More important and surprising:

**Theorem**

(i) *Let $p$ be a sequentially continuous D-homomorphism of $\mathbb{A}$ into $I$. Then $p$ is a probability measure.*

(ii) *Let $h$ be a sequentially continuous D-homomorphisms of $\mathcal{M}(\mathbb{A})$ into $I$. Then $h$ is a probability integral, i.e., there exists a probability measure $p$ on $\mathbb{A}$ such that, $h(f) = \int f\,dp$, $f \in \mathcal{M}(\mathbb{A})$.*

(iii) *Let $(Y, \mathbb{B})$ be a measurable space. Then each sequentially continuous D-homomorphism on $\mathbb{B}$ into $\mathbb{A}$ can be uniquely extended to a sequentially continuous D-homomorphism on $\mathcal{M}(\mathbb{B})$ into $\mathcal{M}(\mathbb{A})$.*

# Observation

If $\{a\}$ is a singleton and $\mathbb{T} = \{\emptyset, \{a\}\}$ is the corresponding trivial field of all subsets of $\{a\}$, then $I$ and $\mathcal{M}(\mathbb{T})$ coincide.

## Theorem

*Let $S$ be a linearly ordered $\sigma$-complete D-poset. Then the following are equivalent:*

(i) *$S$ and $I$ are isomorphic;*

(ii) *$S$ is totally non-atomic;*

(iii) *$S$ is divisible;*

(iv) *If $T$ is a linearly ordered, $\sigma$-complete and divisible D-poset and $S$ is a sub-D-poset of $T$, then $S = T$.*

# Next step: Fuzzification of a cogenerator

BOLD ALGEBRA

- A system $\mathcal{X} \subseteq [0,1]^X$
  containing the constant functions $0_X$, $1_X$ and closed with respect to the complement and (Łukasiewicz) operations $\oplus$, $\odot$: for $a, b \in \mathcal{X}$ put
  $(a \oplus b)(x) = a(x) \oplus b(x) = \min\{1, a(x) + b(x)\}$,
  $(a \odot b)(x) = a(x) \odot b(x) = \max\{0, a(x) + b(x) - 1\}$, $x \in X$.
- MV-algebra representable as $[0,1]$-valued function.
- MV-algebras generalize Boolean algebras,
  bold algebras generalize in a natural way fields of sets.
- Also the unit interval $I = [0,1]$ can be considered as a bold algebra of all measurable $[0,1]$-valued functions.
- Sequentially closed bold algebra $\mathcal{X} \subseteq [0,1]^X$ in $[0,1]^X$ (with respect to the coordinatewise sequential convergence) = **Łukasiewicz tribe**.

# Properties of probability domains

1. ID-posets model the sure event, the impossible event, and the negation of an event. The structure of events is determined by states.
2. Sequentially closed ID-posets satisfy a natural requirement: the probability domains should be closed with respect to sequential limits.
3. Lattice ID-posets are bold algebras.
4. Closed lattice ID-posets are Łukasiewicz tribes.
5. The transition from the classical random events represented by $\sigma$-fields to fuzzy random events represented by measurable functions is characterized by "divisibility".

# Properties of probability domains

6. Łukasiewicz tribes form a category in which both classical and fuzzy events live and the probability of an event can be calculated via an integral.
   From $\mathbb{A}_{\mathcal{X}} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbb{A}_{\mathcal{X}})$ it follows that the classical events ($\sigma$-fields of sets) are "minimal" and the fuzzy events (generated Łukasiewicz tribes) are "maximal" probability domains having nice properties.

# Divisibility

**Definition**

Let $A$ be a D-poset and let $n$ be a natural number, $n > 1$. Assume that for each $a \in A$, $a \neq 0$, there exists an element $a(n) \in A$ such that $0 < a(n) < a$ and, for each $k = 1, 2, \ldots, n-1$, we can subtract from $a$ successively $k$-times $a(n)$ and the result is greater or equal to $a(n)$, and if we subtract from $a$ successively $n$-times $a(n)$, then the result is 0.
Then $A$ is said to be **divisible by $n$**. If $A$ is divisible by $n$ for each natural number $n$, $n > 1$, then $A$ is said to be **divisible**.

**Definition**

Let $A$ be a D-poset and let $B$ be a sub-D-poset of $A$. Assume that for each countable set $S \subset B$ there exists the supremum $\sup S$ of $S$ in $A$ and $\sup S \in B$. Then $B$ is said to be **$\sigma$-complete** in $A$.

# Definition of fuzzification

**Lemma**

*Let $B$ be a sub-D-poset of $I$. If $B$ is divisible and $\sigma$-complete in $I$, then $B = I$.*

**Corollary**

*$I$ is the smallest of all sub-D-posets $B$ of $I$ such that $B$ is divisible and $\sigma$-complete in $I$.*

**Definition**

Let $A$ be a $\sigma$-complete divisible D-poset and let $B$ be a sub-D-poset of $A$. Let $\mathcal{A}$ be the smallest of all sub-D-posets $C$ of $A$ such that

(i) $B$ is a sub-D-poset of $C$;
(ii) $C$ is divisible;
(iii) $C$ is $\sigma$-complete in $A$.

Then $\mathcal{A}$ is said to be a **fuzzification** of $B$.

# Fuzzification of objects

- $\mathbb{A}$ ... a (reduced) $\sigma$-field of subsets of a set $X$
- $\mathcal{M}(\mathbb{A})$ ... the set of all measurable functions ranging in $[0,1]$
- NOTE: $\mathbb{A}$ and $\mathcal{M}(\mathbb{A})$ are Łukasiewicz tribes and D-posets of fuzzy sets

**Lemma**

*The sub-D-poset $\mathcal{M}(\mathbb{A})$ of $[0,1]^X$ is divisible and $\sigma$-complete in $[0,1]^X$.*

**Lemma**

*Let $\mathcal{X}$ be a sub-D-poset of $[0,1]^X$ such that $\mathbb{A} \subseteq \mathcal{X}$. If $\mathcal{X}$ is divisible and $\sigma$-complete in $[0,1]^X$, then $\mathcal{M}(\mathbb{A}) \subseteq \mathcal{X}$ and the sub-D-poset $\mathcal{M}(\mathbb{A})$ is $\sigma$-complete in $\mathcal{X}$.*

# Fuzzification of objects

**Corollary**

*$\mathcal{M}(\mathbb{A})$ is the smallest of all sub-D-posets $\mathcal{X}$ of $[0,1]^X$ such that $\mathbb{A}$ is contained in $\mathcal{X}$ and $\mathcal{X}$ is divisible and $\sigma$-complete in $[0,1]^X$.*

$\Rightarrow$ $\mathcal{M}(\mathbb{A})$ is a fuzzification of $\mathbb{A}$

# Transition from CP to FP – definitions

- $(\Omega, \mathbb{A}, P)$, $(\Xi, \mathbb{B}, Q)$ ... classical probability spaces
- $h$ ... a sequentially continuous D-homomorphisms of $\mathbb{B}$ into $\mathbb{A}$ such that $Q(B) = P(h(B))$ for all $B \in \mathbb{B}$

: $h =$ **measure preserving** $=$ **classical observable**

- $\mathcal{M}(\mathbb{A})$ ... the corresponding ID-poset of measurable fuzzy sets
- $\int (\cdot)\, \mathrm{d}P$ ... the probability integral with respect to $P$

: $(\Omega, \mathcal{M}(\mathbb{A}), \int (\cdot)\, \mathrm{d}P) =$ **fuzzy probability space**

- $(\Omega, \mathcal{M}(\mathbb{A}), \int (\cdot)\, \mathrm{d}P)$, $(\Xi, \mathcal{M}(\mathbb{B}), \int (\cdot)\, \mathrm{d}Q)$ ... fuzzy probability spaces
- $h$ ... a sequentially continuous D-homomorphisms of $\mathcal{M}(\mathbb{B})$ into $\mathcal{M}(\mathbb{A})$ such that $\int v\, \mathrm{d}Q = \int h(v)\, \mathrm{d}P$ for each $v \in \mathcal{M}(\mathbb{B})$

: $h =$ **probability integral preserving** $=$ **fuzzy observable**

: $h =$ **restricted fuzzy observable** if $h(B) \in \mathbb{A}$ for all $B \in \mathbb{B}$

# Transition from CP to FP – categorical aspects

CP ... the category having classical probability spaces as objects and measure preserving classical observables as morphisms
FP ... the category having fuzzy probability spaces as objects and probability integral preserving fuzzy observables as morphisms
RFP ... the subcategory of FP having the fuzzy probability spaces as objects and the restricted fuzzy observables as morphisms

**QUESTION: How are the categories CP and FP related?**

**Theorem**

*Let $(\Omega, \mathbb{A}, P)$ and $(\Xi, \mathbb{B}, Q)$ be classical probability spaces and let $(\Omega, \mathcal{M}(\mathbb{A}), \int(\cdot)\,\mathrm{d}P)$ and $(\Xi, \mathcal{M}(\mathbb{B}), \int(\cdot)\,\mathrm{d}Q)$ be the corresponding fuzzy probability spaces. Let $h_c$ be a classical observable. Then there exists a unique fuzzy observable $h$ such that $h_c(B) = h(B)$ for all $B \in \mathbb{B}$.*

# Transition from CP to FP – SUMMARY

**Theorem**

*The categories CP and RFP are isomorphic.*

**ANSWER:** There is a canonical isomorphism between CP, representing the classical probability theory, and the subcategory RFP of FP, representing the fuzzy probability theory. The objects of the two categories are in a canonical one-to-one correspondence, but the fuzzy probability theory has "more" morphisms: a fuzzy observable can map a crisp random event to a genuine fuzzy random event.