LINEAR TRANSFORMS, BASES and QUADRATIC OPTIMALITY CRITERIA for GAMES

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- As a consequence, the inverse problem for games (find all games having the same Shapley value) has been solved in a tedious way.
- The Shapley value is an example of a least square value as it optimizes some least square criterion on games.
- **Aim of the paper:** to give a systematic analysis of the above aspects.

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- scalar product $\langle v, w \rangle = \sum_{S \subseteq N} v(S)w(S)$

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- To a basis $(f_S)_{S\in\mathcal{N}}$, we make correspond the matrix $F = [f_s]$ of row vectors f_S. Hence $v = \sum_{S\in\mathcal{N}} w_S f_S = wF$ is the expression of v in this basis.

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Lemma (Equivalence between bases and transforms)

For every basis F, there is a (unique) transform Ψ such that for any $v \in \mathbb{R}^{\mathcal{N}}$,

$$v = \sum_{S\in\mathcal{N}} \Psi^v(S) f_S, \tag{1}$$

whose inverse Ψ^{-1} is given by $v \mapsto (\Psi^{-1})^v = \sum_{S\in\mathcal{N}} v(S) f_S = vF$. Conversely, to any transform Ψ corresponds a unique basis F such that (1) holds, given by $f_S = (\Psi^{-1})^{\delta_S}$.

Examples

The Möbius transform: associated with the basis of unanimity games

$$v(S) = \sum_{T \in \mathcal{N}} m^v(T)\zeta_T(S) = \sum_{T \subseteq S} m^v(T), \quad (T \subseteq N),$$

with

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The co-Möbius (or commonality) transform:

$$\check{m}^v(S) = \sum_{T \supseteq N \setminus S} (-1)^{n-|T|} v(T) = \sum_{T \subseteq S} (-1)^{|T|} v(N \setminus T) \quad (S \in \mathcal{N})$$

and $v(S) = \sum_{T \subseteq N \setminus S} (-1)^{|T|} \check{m}^v(T)$.

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and $v(S) = \sum_{T \subseteq N \setminus S} (-1)^{|T|} \check{m}^v(T)$. By the Lemma, the associated basis is

$$f_T(S) = \sum_{B \subseteq N \setminus S} (-1)^{|B|} \delta_T(B) = \begin{cases} (-1)^{|T|} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Examples

The (Shapley) interaction transform:

$$I^v(S) = \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!t!}{(n-s+1)!} \sum_{L \subseteq S} (-1)^{|S \setminus L|} v(T \cup L)$$

and the inverse relation

$$v(S) = \sum_{K \subseteq N} \beta^{|K|}_{|S \cap K|} I^v(K),$$

where

$$\beta^l_k = \sum_{j=0}^{k} \binom{k}{j} B_{l-j} \qquad (k \leq l),$$

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and $B_0, B_1, \ldots$ are the Bernoulli numbers. The associated basis $\{b^I_T\}_{T\in\mathcal{N}}$ is

$$b^I_T(S) = \beta^{|T|}_{|T\cap S|} \quad (S \in \mathcal{N})$$

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The Banzhaf interaction transform:

$$I_{\mathrm{B}}^{v}(S) = \left(\frac{1}{2}\right)^{n-s} \sum_{K \subseteq N} (-1)^{|S \setminus K|} v(K)$$

with inverse relation

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The associated basis $\{b_T^{I_B}\}_{T \in \mathcal{N}}$ is

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It is an orthogonal basis. The corresponding Walsh transform W satisfies

$$v(S) = \sum_{T\subseteq N} W^v(T)(-1)^{|T\setminus S|} \quad (S \in \mathcal{N}),$$

which yields

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Relation between the Hadamard basis and the Walsh basis:

$$b_T^H(S) = b_S^H(T) = \frac{1}{2^{n/2}}(-1)^{|S\cap T|} = \frac{1}{2^{n/2}}(-1)^{|S\setminus(N\setminus T)|} = \frac{1}{2^{n/2}}w_S(N\setminus T)$$

The inverse problem

- A *linear value* is a mapping $\Phi : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{N}$ assigning to any game a n-dim vector. Examples: the Shapley value Φ^{Sh}, the Banzhaf value Φ^{B}.

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- **Fact:** the Shapley (resp., Banzhaf) interaction transform extends the Shapley (resp., Banzhaf) value in the sense that

$$\Phi_i^{\mathrm{Sh}}(v) = I^v(\{i\}), \quad \Phi_i^{\mathrm{B}}(v) = I_B^v(\{i\}), \quad (i \in N)$$

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- Observe that v' is a solution iff $\Phi(v - v') = 0$, i.e., $v - v' \in \ker(\Phi)$. So its suffices to determine the kernel of the linear map Φ.

The inverse problem: Solution 1

Suppose you know a transform Ψ extending the value. Then the kernel is just the space spanned by the vectors f_S of the corresponding basis with $|S| > 1$.

$$v = \sum_{S \in \mathcal{N}} I^v(S) b_S^I = \sum_{i \in N} \Phi_i^{\mathrm{Sh}}(v) b_{\{i\}}^I + \sum_{|S| \neq 1} I^v(S) b_S^I,$$

which implies

$$v \in \ker(\Phi^{\mathrm{Sh}}) \quad \Longleftrightarrow \quad v = \sum_{|S| \neq 1} I^v(S) b_S^I$$

i.e.,

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Illustration with the Shapley value:

$$v = \sum_{S \in \mathcal{N}} I^v(S) b^I_S = \sum_{i \in N} \Phi^{\mathrm{Sh}}_i(v) b^I_{\{i\}} + \sum_{|S| \neq 1} I^v(S) b^I_S,$$

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The inverse problem: Solution 2

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- Put $b_j^\Phi = b_j - \sum_{i=1}^k \epsilon_i^{(j)} b_i$ for $j = k+1, \ldots, 2^n$.

Theorem

Let $B^\Phi = \{b_1, \ldots, b_k, b_{k+1}^\Phi, \ldots, b_{2^n}^\Phi\}$. *Then*

(i) B^Φ *is a basis for* $\mathbb{R}^{\mathcal{N}}$.

(ii) $B_0^\Phi = \{b_{k+1}^\Phi, \ldots, b_{2^n}^\Phi\}$ *is a basis for* $\ker \Phi$.

Least square values

- A *least square value* Φ is given by the solution of a least square optimization problem

$$\min_{x \in \mathbb{R}^N} \sum_{S \in \mathcal{N}} \alpha_S (v(S) - x(S))^2 \quad \text{s.t.} \quad x(N) = v(N)$$

for given coefficients $\alpha_S, S \in \mathcal{N}$, and the convention $x(S) = \sum_{i \in S} x_i$. Then $\Phi_i(v) = x_i^*$, $i \in N$.

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- Well-known fact 1: the Banzhaf value is the solution of the above unweighted ($\alpha_S=1,\forall S$) unconstrained problem (Hammer and Holzman 1987).
- Well-known fact 2: the Shapley value is the solution of the above problem with

$$\alpha_S=\alpha_s=\frac{(n-2)!}{(s-1)!(n-1-s)!} \qquad (s=|S|).$$

(Charnes et al., 1988)

Least square values

- It can be shown that the above problem reduces to

$$\min_{x \in \mathbb{R}^N} xQx^T - xc^T \quad \text{s.t.} \quad x1 = g(v)$$

with $q_{ij} = \sum_{S \ni i,j} \alpha_S$ and $c_i = \sum_{S \ni i} \alpha_S v(S)$. It has always a solution, which is unique iff Q is positive definite.

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Theorem

If Q is regular and positive definite, the (unique) optimal solution x^* is given by:

$$\begin{aligned} z^* &= (2(q+(n-1)p)g - C)/n \quad (\textit{with } C = c\mathbf{1}^T = \textstyle\sum_{i \in N} c_i) \\ x_i^* &= (c_i + z^* - 2pg)/(2q - 2p) \quad (i \in N). \end{aligned}$$