On Kite Pseudo BL-Algebras

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Pseudo BL-algebras

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- (i) $(M; \odot, 1)$ is a monoid (not neces. comm.), $\odot$ is associative with neutral element 1.
  (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.
  (iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ $x, y \in M$.
  (iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x),\ x, y \in M.$
  (v) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow y),\ x, y \in M.$

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- Is every pseudo BL-algebra good ?
- Every pseudo MV-algebra is good

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- Theorem: $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ - pseudo BL-algebra

Properties of Kites

- **Lemma 0.1** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$.

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- **Lemma 0.3** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$*.*
- **Lemma 0.4** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a good pseudo BL-algebra if and only if* $\lambda(J) = \rho(J)$*.*

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- **Lemma 0.5** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$.
- **Lemma 0.6** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a good pseudo BL-algebra if and only if* $\lambda(J) = \rho(J)$.
- There are pseudo BL-algebras that are not good

Properties of Kites

- **Lemma 0.7** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$*.*
- **Lemma 0.8** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a good pseudo BL-algebra if and only if* $\lambda(J) = \rho(J)$*.*
- There are pseudo BL-algebras that are not good
- If G is an Abelian ℓ-group, $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ can be noncommutative

Examples of kites

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- if $I = 2$, $J = 1$, $\lambda(0) = 0$, and $\rho(0) = 1$, we get an algebra $K_{2,1}^{\lambda,\rho}(\mathbf{G})$, Jipsen-Montagna

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- if $I = 2$, $J = 1$, $\lambda(0) = 0$, and $\rho(0) = 1$, we get an algebra $K_{2,1}^{\lambda,\rho}(\mathbf{G})$, Jipsen-Montagna
- $K_{2,1}^{\lambda,\rho}(\mathbb{Z})$ is not good
- $K_{n+1,n}^{\lambda,\rho}(\mathbb{Z})$ with $\lambda(i) = i$ and $\rho(i) = i + 1$, for an arbitrary n

Subdirectly irreducible kites

- **Theorem 0.9** *Let* G *be an* ℓ*-group, and* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *a kite. The following are equivalent:*

1. $\mathbf{G}$ *is subdirectly irreducible and for all* $i, j \in I$ *there exists* $m \in \omega$ *such that* $(\rho \circ \lambda^{-1})^m(i) = j$ *or* $(\lambda \circ \rho^{-1})^m(i) = j$*.*
2. $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *is subdirectly irreducible.*

Subdirectly irreducible kites

- **Theorem 0.11** *Let* $\mathbf{G}$ *be an* ℓ*-group, and* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *a kite. The following are equivalent:*
  1. $\mathbf{G}$ *is subdirectly irreducible and for all* $i, j \in I$ *there exists* $m \in \omega$ *such that* $(\rho \circ \lambda^{-1})^m(i) = j$ *or* $(\lambda \circ \rho^{-1})^m(i) = j$.
  2. $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *is subdirectly irreducible.*
- **Lemma 0.12** *Let* $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ *be a subdirectly irreducible kite. Then,* I *and* J *are at most countably infinite.*

- **Lemma 0.13** *If* G *is an* ℓ*-group,* $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *a subdirectly irreducible kite, and* I *and* J *are finite, then* $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *is isomorphic to one of:*

1. $K^{\emptyset,\emptyset}_{0,0}(\mathbf{G})$, $K^{id,id}_{1,1}(\mathbf{G})$, $K^{\emptyset,\emptyset}_{1,0}(\mathbf{G})$,
2. $K^{\lambda,\rho}_{n,n}(\mathbf{G})$, *for* $n > 1$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1 \pmod{n}$,
3. $K^{\lambda,\rho}_{n+1,n}(\mathbf{G})$, *for* $n > 1$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.

- $I = J = \mathbb{Z}$ $\lambda(i) = i$, $\rho(i) = i + 1$. $K_{\mathbb{Z},\mathbb{Z}}^{\lambda,\rho}(\mathbb{Z})$ is s.i.

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- Take $I = J = \omega$ and put $\lambda(i) = i$, $\rho(i) = i + 1$. The kite $K^{\lambda,\rho}_{\mathbb{Z},\mathbb{Z}}(\mathbb{Z})$ is s.i.

- $I = J = \mathbb{Z}$ $\lambda(i) = i$, $\rho(i) = i + 1$. $K^{\lambda,\rho}_{\mathbb{Z},\mathbb{Z}}(\mathbb{Z})$ is s.i.
- Take $I = J = \omega$ and put $\lambda(i) = i$, $\rho(i) = i + 1$. The kite $K^{\lambda,\rho}_{\mathbb{Z},\mathbb{Z}}(\mathbb{Z})$ is s.i.
- Take $I = J = \omega$ and put $\lambda(i) = i + 1$, $\rho(i) = i$. We obtain an example which is symmetric, but not isomorphic, to the previous one.

- **Theorem 0.14** *Let $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ be a subdirectly irreducible kite. $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ is isomorphic to:*

0. $K_{0,0}^{\emptyset,\emptyset}(\mathbf{G})$, $K_{1,1}^{id,id}(\mathbf{G})$, $K_{1,0}^{\emptyset,\emptyset}(\mathbf{G})$,
1. $K_{n,n}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1 \pmod n$.
2. $K_{\mathbb{Z},\mathbb{Z}}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.
3. $K_{\omega,\omega}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.
4. $K_{\omega,\omega}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j + 1$ *and* $\rho(j) = j$.
5. $K_{n+1,n}^{\lambda,\rho}(\mathbf{G})$, *with* $\lambda(j) = j$ *and* $\rho(j) = j + 1$.

- **Corollary 0.15** *A subdirectly irreducible kite is good if and only if it is either a pseudo MV-algebra or it is* $K^{\emptyset,\emptyset}_{1,0}(\mathbf{G})$*, for a subdirectly irreducible* ℓ*-group* $\mathbf{G}$*.*

- **Corollary 0.18** *A subdirectly irreducible kite is good if and only if it is either a pseudo MV-algebra or it is* $K_{1,0}^{\emptyset,\emptyset}(\mathbf{G})$, *for a subdirectly irreducible* ℓ*-group* $\mathbf{G}$.
- **Theorem 0.19** *Every kite is subdirectly embeddable into a product of subdirectly irreducible kites.*

- **Corollary 0.21** *A subdirectly irreducible kite is good if and only if it is either a pseudo MV-algebra or it is* $K_{1,0}^{\emptyset,\emptyset}(\mathbf{G})$*, for a subdirectly irreducible* ℓ*-group* $\mathbf{G}$*.*
- **Theorem 0.22** *Every kite is subdirectly embeddable into a product of subdirectly irreducible kites.*
- **Theorem 0.23** *For any integer* $n \geq 0$*,* $V(\mathbb{Z}_n^{\dagger})$ *is a cover of the variety* BA*, and* $n \neq m$ *implies* $V(\mathbb{Z}_n^{\dagger}) \neq V(\mathbb{Z}_m^{\dagger})$*.*

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Thank you for your attention