The General Nilpotent System

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Motivation and background
Motivation and background

**Nilpotent Connective Systems**

**Fuzzy set theory**
Proper choice of fuzzy connectives

**Nilpotent operators**
Nilpotent t-norms and t-conorms: preferable properties

**Operators – Systems**
Instead of pure operators, we examine connective SYSTEMS

**Our results**
Consistent nilpotent connective systems which are not isomorphic to Łukasiewicz system
Basic preliminaries
**Negations**

**Definition**
A decreasing function $n(x) : [0, 1] \rightarrow [0, 1]$ is a fuzzy negation, if $n(0) = 1$ and $n(1) = 0$.

**Definition**
A fuzzy negation is
- **strict**, if it is strictly decreasing and continuous
- **strong**, if it is involutive, i.e. $n(n(x)) = x$ for $\forall x \in [0, 1]$.

**Trillas’ Theorem**

$n(x)$ is a strong negation if and only if

$$n(x) = f_n(x)^{-1}(1 - f_n(x)),$$

where $f_n(x) : [0, 1] \rightarrow [0, 1]$ is continuous and strictly increasing.
**Triangular norms and conorms**
(Schweizer and Sklar, 1960)

**T-norm $T$ (Conjunction)**

A t-norm $T$ is a function on $[0, 1]^2$ that satisfies, for all $x, y, z \in [0, 1]$:
- $T(x, 1) = x$ (neutral element 1);
- $x \leq y \Rightarrow T(x, z) \leq T(y, z)$ (monotonicity);
- $T(x, y) = T(y, x)$ (commutativity);
- $T(T(x, y), z) = T(x, T(y, z))$ (associativity).

**T-conorm $S$ (Disjunction)**

- $S(x, 0) = x$ (neutral element 0) plus the other three properties above.
A continuous t-norm $T$ is said to be
- **Archimedean** if $T(x, x) < x$ holds for all $x \in (0, 1)$,
- **strict** if $T$ is strictly monotone i.e. $T(x, y) < T(x, z)$ whenever $x \in [0, 1]$ and $y < z$, and
- **nilpotent** if there exist $x, y \in (0, 1)$ such that $T(x, y) = 0$.

A continuous t-conorm $S$ is said to be
- **Archimedean** if $S(x, x) > x$ holds for every $x, y \in (0, 1)$,
- **strict** if $S$ is strictly monotone i.e. $S(x, y) < S(x, z)$ whenever $x \in [0, 1]$ and $y < z$, and
- **nilpotent** if there exist $x, y \in (0, 1)$ such that $S(x, y) = 1$. 
**Additive generators**

A function \( T : [0, 1]^2 \rightarrow [0, 1] \) is a continuous Archimedean t-norm iff it has a continuous additive generator, i.e. there exists a continuous strictly decreasing function \( t : [0, 1] \rightarrow [0, \infty] \) with \( t(1) = 0 \), which is uniquely determined up to a positive multiplicative constant, such that

\[
T(x, y) = t^{-1}(\min(t(x) + t(y), t(0)), \quad x, y \in [0, 1]. \tag{1}
\]

A function \( S : [0, 1]^2 \rightarrow [0, 1] \) is a continuous Archimedean t-conorm iff it has a continuous additive generator, i.e. there exists a continuous strictly increasing function \( s : [0, 1] \rightarrow [0, \infty] \) with \( s(0) = 0 \), which is uniquely determined up to a positive multiplicative constant, such that

\[
S(x, y) = s^{-1}(\min(s(x) + s(y), s(1)), \quad x, y \in [0, 1]. \tag{2}
\]
A t-norm $T$ is \textit{strict} if and only if $t(0) = \infty$.

A t-norm $T$ is \textit{nilpotent} if and only if $t(0) < \infty$.

A t-conorm $S$ is \textit{strict} if and only if $s(1) = \infty$.

A t-conorm $S$ is \textit{nilpotent} if and only if $s(1) < \infty$. 
Nilpotent operators

Preferable Properties

Law of contradiction $c(x, n(x)) = 0$

Excluded middle $d(x, n(x)) = 1$

Coincidence of residual and S-implications...

\[
T_L(x, y) = \max(x + y - 1, 0)
\]

\[
S_L(x, y) = \min(x + y, 1)
\]
Normalized generator functions — uniquely determined

\[ f_c(x) := \frac{t(x)}{t(0)}, \quad f_d(x) := \frac{s(x)}{s(1)}. \]

\[ f_c(x), f_d(x), f_n(x) : [0, 1] \to [0, 1]. \]

Cutting operation

\[ [x] = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } 0 \leq x \leq 1 \\ 1 \text{ if } 1 < x \end{cases} \]
**Cutting function**

\[ c(x, y) = f_c^{-1}[f_c(x) + f_c(y)] \]
\[ d(x, y) = f_d^{-1}[f_d(x) + f_d(y)] \]

**Min**

\[ \text{Min}(x, y) = f_c^{-1} [f_c(x) + [f_c(y) - f_c(x)]] \]

**Max**

\[ \text{Max}(x, y) = n \left( f_c^{-1} [f_c(n(x)) + [f_c(n(y)) - f_c(n(x))]] \right) \]

\[ T_L(x, y) = [x + y - 1] \]
\[ S_L(x, y) = [x + y] \]
Connective Systems
**Connective Systems**

- **strong negation** $n(x)$
- **conjunction** $c(c, y)$
- **disjunction** $d(x, y)$

**Definition**
The triple $(c(x, y), d(x, y), n(x))$, where $c(x, y)$ is a t-norm, $d(x, y)$ is a t-conorm and $n(x)$ is a strong negation, is called a **connective system**.

**Definition**
A connective system is **nilpotent**, if the conjunction is a nilpotent t-norm, and the disjunction is a nilpotent t-conorm.
**Definition**

Two connective systems \((c_1(x, y), d_1(x, y), n_1(x))\) and \((c_2(x, y), d_2(x, y), n_2(x))\) are **isomorphic**, if there exists a bijection \(\phi : [0, 1] \to [0, 1]\) such that

\[
\phi^{-1}(c_1(\phi(x), \phi(y))) = c_2(x, y)
\]

\[
\phi^{-1}(d_1(\phi(x), \phi(y))) = d_2(x, y)
\]

\[
\phi^{-1}(n_1(\phi(x))) = n_2(x)
\]

**Definition**

A connective system is called **Łukasiewicz** system, if it is isomorphic to \(([x + y - 1], [x + y], 1 - x)\), i.e. it has the form \((\phi^{-1}[\phi(x) + \phi(y) - 1], \phi^{-1}[\phi(x) + \phi(y)], \phi^{-1}[1 - \phi(x)])\).
**Classification Property:**

- **law of contradiction:**
  \[ c(x, n(x)) = 0, \]

- **excluded middle:**
  \[ d(x, n(x)) = 1. \]

**De Morgan Laws:**

\[ c(n(x), n(y)) = n(d(x, y)) \]

or

\[ d(n(x), n(y)) = n(c(x, y)). \]
IS IT SENSIBLE TO USE MORE THAN ONE GENERATOR FUNCTIONS?
Results
Among nilpotent systems, Łukasiewicz connective system \((T_L, S_L, \text{standard negation})\) is characterized by \(f_c(x) + f_d(x) = 1, \quad \forall x \in [0, 1]\).

For \(f_c(x) + f_d(x) < 1, \quad \forall x \in [0, 1]\): the system is not consistent.

For \(f_c(x) + f_d(x) > 1, \forall x \in [0, 1]\): BOUNDED SYSTEMS
**Bounded Systems**

**Three different negations**

\[
\begin{align*}
n_d(x) &= f_d^{-1}(1 - f_d(x)) < n(x) < n_c(x) = f_c^{-1}(1 - f_c(x)). \\

f_c(x) + f_d(x) &= 1 \iff n_c(x) = n(x) = n_d(x)
\end{align*}
\]
Theorem

In a connective structure the *classification property* holds if and only if

\[ n_d(x) \leq n(x) \leq n_c(x), \quad \forall x \in [0, 1]. \]
**THEOREM**

In a connective structure the *classification property* holds if and only if

\[ n_d(x) \leq n(x) \leq n_c(x), \quad \forall x \in [0, 1]. \]

**THEOREM**

In a connective system the *De Morgan law* holds if and only if

\[ n(x) = f_c^{-1}(f_d(x)) = f_d^{-1}(f_c(x)), \quad \forall x \in [0, 1]. \]
**Consistency**

1. If \( c(x, y), d(x, y) \) and \( n(x) \) fulfil the De Morgan identity and the classification property (i.e. they form a consistent system), then 
\[
f_c(x) + f_d(x) \geq 1, \quad \forall x \in [0, 1].
\]

2. If \( f_c(x) + f_d(x) \geq 1, \quad \forall x \in [0, 1] \) and the De Morgan law holds, then the classification property also holds (which now means that the system is consistent).
Example for \( f_c(x) + f_d(x) > 1 \)

For \( f_c(x) := 1 - x^\alpha, f_d(x) := 1 - (1 - x)^\alpha, n(x) := 1 - x, \quad \alpha \in (1, \infty) \).

- the connective system is consistent,
- \( f_c(x) + f_d(x) > 1, \quad \forall x \in [0, 1] \)
  (or equivalently \( n_d(x) < n(x) < n_c(x), \quad \forall x \in [0, 1] \)).
RATIONAL GENERATORS

For the Dombi functions (from 'pliant systems')

\[ f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu} \frac{1-x}{x}} \]

\[ f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c} \frac{x}{1-x}} \]

\[ f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d} \frac{1-x}{x}} \]

the following statements are equivalent:

1. The connective structure defined by the Dombi functions satisfies the De Morgan law.

\[ \left( \frac{1-\nu}{\nu} \right)^2 = \frac{\nu_c}{1-\nu_c} \frac{1-\nu_d}{\nu_d}. \]
RATIONAL GENERATORS

For the Dombi functions (from 'pliant systems')

\[ f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu} \frac{1-x}{x}} \]

\[ f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c} \frac{x}{1-x}} \]

\[ f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d} \frac{1-x}{x}} \]

the following statements are equivalent:

1. \( n_d(x) < n(x) < n_c(x) \)
2. \( \nu_d < \nu < \nu_c \)
3. Given that the De Morgan property holds, \( f_c(x) + f_d(x) > 1 \), or equivalently \( \nu_c + \nu_d < 1 \).
Examples

Rational generators

\[ \nu_c = 0.6 \text{ and } \nu_d = 0.4 \]

\[ \nu_c = 0.4 \text{ and } \nu_d = 0.2 \]
Conclusion and further work
Consistent nilpotent connective systems using more than one generator functions (not new operators, new system)

\[ f_c(x) + f_d(x) > 1 \]
\[ n_d(x) < n(x) < n_c(x) \]

2 naturally derived thresholds \((\nu_c, \nu_d)\)

Further work: implication, equivalence
THANK YOU FOR YOUR ATTENTION!

"Life is like riding a bicycle. To keep your balance you must keep moving."

/Einstein/