On fuzzy entropy and topological entropy of fuzzy extensions of dynamical systems

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OP Výzkum
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IRAFM

- X ... a compact metric space
- I ... the unit interval $[0,1]$
- $f \in C(X)$ ... a continuous map $f : X \to X$
- Φ ... the Zadeh's extension of f

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \mathbb{F}(X) & \xrightarrow{\Phi} & \mathbb{F}(X) \end{array}$$

- There exists (X, f) such that

$$h(f) = 0, \quad 0 < h(\bar{f}) < \infty, \quad ent(\Phi) = \infty.$$

- If $X = [0,1]$, then

$$h(f) = ent(\Phi)$$

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For any fuzzy set $A \in \mathbb{F}(X)$ a fuzzy entropy (degree of fuzziness) $e(A)$ can be defined by a function $e : \mathbb{F}(X) \to [0,1]$ satisfying the following axioms, below $A, B \in \mathbb{F}(X)$:

A1. $e(A) = 0$ if and only either $A = \chi_C$ for some $C \subseteq X$ or $A = \emptyset$.

A2. $e(A) = 1$ if and only if $A = \frac{1}{2}\chi_X$.

A3. $e(A) \leq e(B)$ whenever A is less fuzzy than B, that is $A(x) \leq B(x) \leq 1/2$ or $A(x) \geq B(x) \geq 1/2$ for all $x \in X$.

A4. $e(A) = e(A^c)$.

- X denotes a compact metric space
- a **characteristic function** $\chi_B : X \to I$ on $B \subseteq \mathbb{X}$

$$\chi_B(x) = \begin{cases} 1 & x \in B, \\ 0 & x \notin B. \end{cases}$$

- a fuzzy set $A : X \to I$
- for $\alpha \in I$, α-**cut** of A is $[A]_\alpha = \{x \in X \mid A(x) \geq \alpha\}$
- **support** of A is defined by

$$supp(A) = \overline{\{x \in X \mid A(x) > 0\}}.$$

- $\mathbb{F}(X)$ ... the family of **upper semicontinuous** fuzzy sets on X
- $\mathbb{F}^1(X)$ ... the family of **normal** upper-semicontinuous fuzzy sets on X
- $\mathbb{F}^1_c(X)$ ... the family of normal upper-semicontinuous fuzzy **numbers** on X

Let (X,d) denote a (locally) compact metric space and let A, B be nonempty compact subsets of X. The **Hausdorff metric** D_X between A and B is defined by

$$D_X(A,B) = \inf\{\varepsilon > 0 \mid A \in U_\varepsilon(B) \text{ and } B \in U_\varepsilon(A)\},$$

where

$$U_\varepsilon(A) = \{x \in X \mid D(x,A) < \varepsilon\}, \text{ and } D(x,A) = \inf_{a \in A} d(x,a).$$

By $\mathbb{K}(X)$ we denote the metric space of all nonempty compact subsets of X.

It is well known that $\mathbb{K}(X)$ is compact, complete and separable whenever X itself is compact, complete and separable, respectively.

For any $A \in \mathbb{F}(X)$,

$$end(A) = \{(x, a) \in X \times I \mid A(x) \geq a\}$$

and

$$send(A) = end(A) \cap (supp(A) \times I).$$

The **sendograph** metric [P. E. Kloeden, 1982]

$$d_S(A, B) = D_{X \times I}(send(A), send(B))$$

is defined for nonempty fuzzy sets $A, B \in \mathbb{F}_0(X)$ and the **endograph** metric [Fan, 2004] is defined for any two $A, B \in \mathbb{F}(X)$

$$d_E(A, B) = D_{X \times I}(end(A), end(B)).$$

We define the **levelwise** metric [Kaleva, Seikkala, 1984] on $\mathbb{F}_0(X)$ by

$$d_\infty(A, B) = \sup_{\alpha \in I} D_X([A]_\alpha, [B]_\alpha).$$

For a given map $f \in C(X)$, we define its **fuzzification** or **Zadeh's extension** $\Phi : \mathbb{F}(X) \to \mathbb{F}(X)$ by

$$(\Phi(A))(y) = \sup_{x \in f^{-1}(y)} \{A(x)\}.$$

We also define $\bar{f} : \mathbb{K}(X) \to \mathbb{K}(X)$ as a **set-valued extension** of f by

$$\bar{f}(A) = f(A) \text{ for any } A \in \mathbb{K}(X).$$

Let us introduce the Bowen's definition of **topological entropy** for continuous maps ([Bowen]). Let $K \subset X$ be a compact subset and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that a set $E \subset K$ is (n, ε, K, f)-**separated** (by the map f) if for any $x, y \in E$, $x \neq y$, there is $k \in \{0, 1, ..., n-1\}$ such that $d(f^k(x), f^k(y)) > \varepsilon$. Denote by $s_n(\varepsilon, K, f)$ the cardinality of any maximal (n, ε, K, f)-separated set in K and define

$$s(\varepsilon, K, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, K, f).$$

Now the **topological entropy** of f is

$$h_d(f) = \sup_K \lim_{\varepsilon \to 0} s(\varepsilon, K, f).$$

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If the space X is not compact, we use the following definition of topological entropy

$$\mathrm{ent}(f) = \sup\{h(f|_K) : K \in \mathcal{K}_f(X)\},$$

where $\mathcal{K}_f(X)$ denotes the set of f-**invariant** (i.e., $f(A) \subseteq A$ for any $A \subseteq X$) compact subsets of X.

For a continuous map $f : X \to X$ on a compact metric space we denote by $\beta(X)$ the Borel sets of X. Then, a probabilistic measure $\mu : \beta(X) \to [0,1]$ is said to be **invariant** (resp. f-invariant) by f if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \beta(X)$. If we denote by $\mathcal{M}(X, f)$ the set of f-invariant measures on X, it is known that this set is nonempty, compact and convex.

An f-invariant measure is **ergodic** if either $\mu(A) = 1$ or 0 for any invariant set A. The set of ergodic measures on X will be denoted by $\mathcal{E}(X, f)$. It is known that this set is also nonempty and compact in $\mathcal{M}(X, f)$. For a continuous map $\varphi : X \to \mathbb{C}$ and any $x \in X$ and $\mu \in \mathcal{E}(X, f)$ the Birkhoff's Ergodic Theorem states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^n(x) = \int_X \varphi d\mu$$

μ-almost everywhere.

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A **measurable partition** of a probabilistic space X is a family of measurable pairwise disjoint subsets of X whose union is X. Let $\mathcal{A} = \{A_1, ..., A_k\}$ be a finite partition of measurable sets of X. We define the **metric entropy** of the partition $\mathcal{A}$ as

$$H_\mu(\mathcal{A}) = -\sum_{i=1}^{k} \mu\left(A_i\right) \log \mu\left(A_i\right)$$

(here $0 \log 0 = 0$). For given finite partitions $\mathcal{A}$ and $\mathcal{B}$, one can define a new finite partition $\mathcal{A} \vee \mathcal{B} = \{A_i \cap B_j : A_i \in \mathcal{A},\ B_j \in \mathcal{B}\}$. Define the **metric entropy** of f over the partition $\mathcal{A}$

$$h_\mu(f, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{A}\right).$$

Then, the **metric entropy** (also called **measure theoretical entropy**) of f is the non-negative number

$$h_\mu\left(f\right) = \sup_{\mathcal{A}} h_\mu(f, \mathcal{A}).$$

According to [Knopfmacher], we need a measure space $(X, \beta(X), \mu)$, where $\beta(X)$ is the Borel σ-algebra, μ is a nonzero finite measure, and $\mathbb{F}(X)$ consists of μ-measurable functions. Then, for any real-valued function $g : I \to \mathbb{R}$ such that

- $g(0) = g(1) = 0$,
- $g(\alpha) = g(1 - \alpha)$ for any $\alpha \in I$,
- g is strictly increasing on $[0, 1/2]$,

the expression

$$e_\mu(A) = \frac{1}{\mu(X)} \int g(A(x)) d\mu(x) \tag{1}$$

defines a **degree of fuzziness** $e(A)$ of a fuzzy set $A \in \mathbb{F}(X)$. Moreover, if $A = \sum_{i=1}^{k} a_i \chi_{X_i}$, with $a_i \in (0, 1]$ and $X_i \in \beta(X)$, $i = 1, ..., k$, then $e_\mu(A) = \sum_{i=1}^{k} g(a_i)\mu(X_i)$.

Lemma

([Knopfmacher, 1975]) *The degree of fuzziness* $e : \mathbb{F}(X) \to \mathbb{R}$ *has the following properties* ($A \in \mathbb{F}(X)$)*:*

A1. $e(A) = 0$ *if and only either* $A \overset{a.e.}{=} \chi_C$ *for some* $C \subseteq X$ *or* $A \overset{a.e.}{=} \emptyset_X$,

A2. $e(A)$ *is maximal if and only if* $A \overset{a.e.}{=} \frac{1}{2}\chi_X$,

A3. $e(A) \leq e(B)$ *whenever* A *is less fuzzy than* B, *that is* $A(x) \leq B(x) \leq 1/2$ *or* $A(x) \geq B(x) \geq 1/2$ *for almost all* $x \in X$,

A4. $e(A) = e(A^c)$,

A5. e *is continuous with respect to the supremum metric on* $\mathbb{F}(X)$.

Example

Let μ be a probabilistic measure whose support is a singleton set $\{a_0\}$ and let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence converging to a_0 such that $a_i \neq a_0$ for each $i \in \mathbb{N}$. Then, clearly for some fixed $c \in (0,1)$ and any g as above,

$$e_\mu(c \cdot \chi_{a_0}) > 0,$$

but

$$e_\mu(c \cdot \chi_{a_i}) = 0, \quad \textit{for each } i \in \mathbb{N}.$$

However, $\{c \cdot \chi_{a_i}\}_{i\in\mathbb{N}}$ converges to $c \cdot \chi_{a_0}$ in the metric topology induced by any of d_E, d_S and d_∞, which implies the discontinuity of e_μ.

Theorem

Let $X = I = [0,1]$, λ be the Lebesgue measure λ on the Borel σ-algebra $\beta(I)$ of I and g be continuous. The degree of fuzziness $e_\lambda : \mathbb{F}_c^1(I) \to \mathbb{R}$ is continuous with respect to d_∞.

Let e_μ be a degree of fuzziness, with μ a probability measure on the Borel sets of X. Then, for $A \in \mathbb{F}^1(X)$ we can define the **degree of fuzziness along the orbit** of A as

$$df(A, e_\mu) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} e_\mu \circ \Phi^i(A).$$

If e_μ is continuous and $m \in \mathcal{E}(\mathbb{F}^1(X), \Phi)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} e_\mu \circ \Phi^i(A) = \int_{\mathbb{F}^1(X)} e_\mu dm = m(e_\mu)$$

for almost all $A \in \mathbb{F}^1(X)$.

We define the degree of fuzziness of Φ as

$$df(\Phi, e_\mu) = \sup\{df(A, e_\mu) : A \in \mathbb{F}^1(X)\}.$$

It is easy to see that the degree of fuzziness is preserved by a conjugacy.

If e_μ is continuous, we can define the **ergodic degree of fuzziness** of the map Φ as

$$edf(\Phi, e_\mu) = \sup\{m(e_\mu) : m \in \mathcal{E}(\mathbb{F}^1(X), \Phi)\},$$

which again is a measure of the fuzziness of a continuous map.

Lemma

$edf(\Phi, e_\mu) \leq df(\Phi, e_\mu)$

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \varphi & & \downarrow \varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

Theorem

Let $e_\mu : \mathbb{F}^1(X) \to [0,1]$ *be a degree of fuzziness on* $\mathbb{F}^1(X)$. *Let* $f : X \to X$ *and* $g : Y \to Y$ *be continuous maps on compact metric spaces. If they are conjugated, with conjugacy* φ, *then*

$$edf(\Phi_f, e_\mu) = edf(\Phi_g, e_\nu),$$

where $\nu = \mu \circ \varphi^{-1}$.

For a fixed degree of fuzziness $e_\mu : \mathbb{F}(X) \to [0,1]$ and for $\alpha \in [0,1]$, let

$$\mathcal{F}_\alpha(e_\mu) = \left\{ A \in \mathbb{F}^1(X) : df(A, e_\mu) = \alpha \right\}.$$

Lemma

$\mathcal{F}_\alpha(e_\mu)$ is Φ-invariant.

If e_μ is continuous and $\alpha \in [0,1]$, let

$$\mathcal{E}\mathcal{F}_\alpha(e_\mu) = \left\{ A \in \mathbb{F}^1(X) : edf(A, e_\mu) = \alpha \right\}.$$

Lemma

$\mathcal{E}F_\alpha(e_\mu)$ is Φ-invariant.

Lemma

$\mathcal{E}\mathcal{F}_\alpha(e_\mu) \subset \mathcal{F}_\alpha(e_\mu)$ for any $\alpha \in [0,1]$.

We define the **dynamic fuzzy entropy** of Φ, denoted by $\mathrm{fuzzent}(\Phi)$ in the following way

$$\mathrm{fuzzent}(\Phi) = \mathrm{ent}(\Phi|_{\bigcup_{\alpha\in(0,1]}\mathcal{F}_\alpha(e_\mu)}) \tag{2}$$

If e_μ is continuous, we can make use of ergodic measures and the variational principle for topological entropy to define a new notion which we call **ergodic fuzzy entropy** by

$$\mathrm{efuzzent}(\Phi) = \sup\left\{h_m(\Phi) : m \in \mathcal{E}(\mathbb{F}^1(X), \Phi) \ \text{ and } \ m(e_\mu) > 0\right\}. \tag{3}$$

Theorem

The following statements hold:

1. *If two continuous maps* $f : X \to X$ *and* $g : Y \to Y$ *are conjugate, then* $\mathrm{fuzzent}(\Phi_f) = \mathrm{fuzzent}(\Phi_g)$.
2. *For any* $n \in \mathbb{N}$ *we have that* $\mathrm{fuzzent}(\Phi^n) = n \cdot \mathrm{fuzzent}(\Phi)$.
3. *If* Φ *is a homeomorphism, then* $\mathrm{fuzzent}(\Phi) = \mathrm{fuzzent}(\Phi^{-1})$.

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Theorem

Let $f : X \to X$ be continuous and let $\Phi : \mathbb{F}^1(X) \to \mathbb{F}^1(X)$ be its Zadeh's extension. Let e_μ be a continuous degree of fuzziness. Then

$$\mathrm{efuzzent}(\Phi) \geq \mathrm{fuzzent}(\Phi). \tag{4}$$

Example

Let $X = I = [0, 1]$ and let $f : I \to I$ be continuous. Let $\Phi : \mathbb{F}^1_c(I) \to \mathbb{F}^1_c(I)$ be the Zadeh's extension of f on the fuzzy numbers of I. Let λ be the Lebesgue measure on I and let e_λ be the degree of fuzziness generated by λ. We get $\mathrm{fuzzent}(\Phi) = h(f)$.

- Several instruments for dealing with degree of fuzziness in fuzzy dynamical systems were elaborated.
- Some basic properties were studied.
- Although some questions are left open ...
- We demonstrated that in some special (but still reasonable) cases it is sufficient to deal with fuzzy sets with zero degree of fuzziness.

Thanksgiving

Thank You for Your Attention