Fuzzy logic and the similarity-based interpretation of fuzzy sets

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Jan. 2012

Our topic:

Reasoning under vagueness

We intend to formalise
reasoning about properties
which we express in natural language,
without using any measuring devices.

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  do not require special methods
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The challenge of vagueness:

- Combining a coarse and a fine level of granularity:
  Finding a reasoning system for reasoning with
  qualitative information in a quantitative framework.

A simple approach

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a **set of worlds** W.

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The qualitative information:
for each vague property, its set of **prototypes** in W.

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- We consider implications of the form:

$$\alpha_1, \ldots, \alpha_n \to \beta$$

"A situation well described by α_1 and … and α_n is also well described by β."

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- Contradiction is expressible by the constant $\perp$.

PGL: Positive Gentzenian Logic

Formulas:
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Interpretation:
A model for PGL is a non-empty set W, called a set of worlds.
An evaluation v maps each proposition to a subset of W,
preserving $\wedge, \vee, \bot, \top$.
An implication $\alpha \wedge \beta \to \gamma \vee \delta$ is satisfied by v if

$$v(\alpha) \cap v(\beta) \subseteq v(\gamma) \cup v(\delta).$$

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PGL is the *Logic of Distributive* $0, 1$-*Lattices*.

PGL: a proof system
(Font, Verdú)

$$\bot \to \alpha \qquad \alpha \to \alpha \qquad \alpha \to \top$$

$$\frac{\Gamma \to \alpha}{\Gamma, \beta \to \alpha} \qquad \frac{\Gamma, \alpha, \beta \to \gamma}{\Gamma, \alpha \wedge \beta \to \gamma} \qquad \frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}$$

$$\frac{\Gamma, \alpha \to \gamma \quad \Gamma, \beta \to \gamma}{\Gamma, \alpha \vee \beta \to \gamma} \qquad \frac{\Gamma \to \alpha}{\Gamma \to \alpha \vee \beta} \quad \frac{\Gamma \to \beta}{\Gamma \to \alpha \vee \beta}$$

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A graded approach

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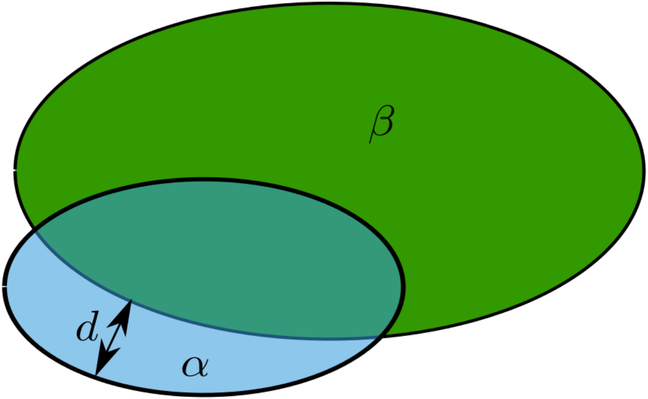
Approximate reasoning
(Ruspini)

We consider graded implications:

$$\alpha \xrightarrow{d} \beta$$



Connectives in approximate reasoning

The statement

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reads as:

"If α and β fit to some degree $\geq t \in [0,1]$,
then γ or δ fit to the degree $\geq t \odot d$."

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But how should we interpret the "and" and the "or" here?

Logics for approximate reasoning

(Godo, Esteva, Rodríguez, Dubois, Prade, ...)

In a (version of) approximate reasoning, we interpret

$$\alpha \wedge \beta \xrightarrow{d} \gamma \vee \delta$$

w.r.t. to an evaluation v as

"If a world is similar to $v(\alpha) \cap v(\beta)$ to the degree t, then to $v(\gamma) \cup v(\delta)$ to the degree $\geq t \odot d$."

Approximate reasoning for reasoning under vagueness

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Limitations:

- (Complete) axiomatisation in important cases not known.
- Rules too weak for certain applications.

Fuzzy-logic-like interpretation of the connectives

We want to understand $\alpha \wedge \beta \xrightarrow{d} \gamma \vee \delta$ as

"If α and β fit to the degree t,
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and interpret it as

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We want to understand $\alpha \wedge \beta \xrightarrow{d} \gamma \vee \delta$ as

"If a situation fits both to α and β to the degree $\geq t$, it fits either γ or δ to the degree $\geq t \odot d$."

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Then:

- implicit connection of truth degrees by min/max.
- much like FL, where degrees are similarities.

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An implication $\alpha \wedge \beta \xrightarrow{d} \gamma \vee \delta$ is satisfied by v if

$$U_t(v(\alpha)) \cap U_t(v(\beta)) \subseteq U_{d \odot t}(v(\gamma)) \cup U_{d \odot t}(v(\delta))$$

for any $t \in [0,1]$.

Rules for gPGL

$$\bot \overset{d}{\to} \alpha \qquad \alpha \overset{d}{\to} \alpha \qquad \alpha \overset{d}{\to} \top \qquad \alpha \overset{0}{\to} \beta$$

$$\frac{\Gamma \overset{d}{\to} \alpha}{\Gamma, \beta \overset{d}{\to} \alpha} \qquad \frac{\Gamma, \alpha, \beta \overset{d}{\to} \gamma}{\Gamma, \alpha \wedge \beta \overset{d}{\to} \gamma} \qquad \frac{\Gamma \overset{d}{\to} \alpha \quad \Gamma \overset{d}{\to} \beta}{\Gamma \overset{d}{\to} \alpha \wedge \beta}$$

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The rules are well sufficient for practical applications like expert systems.
Completeness does not hold, however.

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Furthermore:

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Furthermore:

- gPGL is well-applicable,
  even without a complete proof system.
- Completeness *can* be achieved if restricting to "non-disjunctive" theories.

Alternative style of rules for gPGL

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The easiest possibility is to replace

$$\alpha_1, \ldots, \alpha_n \xrightarrow{d} \beta$$

by

$$(\alpha_1, s_1), \ldots, (\alpha_n, s_n) \to (\beta, t).$$

Negation

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We treat $\neg\varphi$ like φ:
$\neg\varphi$ is modelled by the "prototypes" of $\neg\varphi$,
that is, by the set of constrasting cases of φ.

Models of negated properties

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Otherwise, we have (at least) the following possibilities of choosing an interpretation $v(\neg\varphi) \subseteq W$:

- We construct $v(\neg\varphi)$ from $v(\varphi)$.
- We construct $v(\neg\varphi)$ from all $v(\psi)$, where ψ contradicts φ.
- We assume only that $v(\neg\varphi)$ is disjoint from $v(\varphi)$.
- We choose $v(\neg\varphi)$ freely.

Logics with negation

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  This leads to CPL.
- For ngPGL, the $\frac{1}{2}$-neighborhoods of $v(\varphi)$ and $v(\neg\varphi)$ form a partition.
  This corresponds to the standard negation in FL.

Summary

The Positive Gentzenian Logics ...

- can be based on rules with a clear intuitive meaning.
- are neither fuzzy logics nor logics for approximate reasoning, but something “in between”.
- are well-applicable, even though completeness can be shown only in a restricted sense.