Multidimensional Possibilistic Models

Jiřina Vejnarová

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic

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Outline

1. Background

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Knowledge representation

Two main issues to be solved simultaneously:

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- multidimensionality,
- uncertainty.

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to describe uncertainty of e. g. 250 variables you need at least

$$(2^{250} - 1) \text{ probabilities}$$

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Possibility measure

Possibility measure on $\mathbf{X}$ ($|\mathbf{X}| < \infty$)

$$\Pi : \mathcal{P}(\mathbf{X}) \longrightarrow [0, 1]$$

(i) $\Pi(\emptyset) = 0$;

(ii) for any family $\{A_j, j \in J\}$ of elements of $\mathcal{P}(\mathbf{X})$

$$\Pi(\bigcup_{j \in J} A_j) = \max_{j \in J} \Pi(A_j).$$

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Π is *normal* iff $\Pi(\mathbf{X}) = 1$.

Possibility distribution

Possibility distribution of Π

$$\pi : \mathbf{X} \longrightarrow [0, 1],$$

such that for any $A \in \mathcal{P}(\mathbf{X})$

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Let $\pi(x,y)$ be a possibility distribution on $\mathbf{X} \times \mathbf{Y}$. Its *marginal possibility distribution* on $\mathbf{X}$ is defined by

$$\pi_X(x) = \max_{y \in \mathbf{Y}} \pi(x,y)$$

for any $x \in \mathbf{X}$.

Conditioning

Conditional possibility distribution $\pi_{X|_T Y}$ is defined as *any* solution of the equation

$$\pi_{XY}(x,y) = T\left(\pi_Y(y), \pi_{X|_T Y}(x|_T y)\right)$$

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which means that

$$T\left(\pi_Y(y), \pi_{X|_T Y}(x|_T y)\right) = T\left(\pi_Y(y), \pi_{XY}(x,y) \triangle_T \pi_Y(y)\right).$$

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and, furthermore,

$$\pi_{X|_T Y}(x|_T y) \sqsubseteq \pi_{XY}(x,y) \triangle_T \pi_Y(y).$$

Independence

Variables X and Y are *possibilistically T-independent* (with respect to π) if for any pair $(x, y) \in \mathbf{X} \times \mathbf{Y}$,

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Variables X and Y are *possibilistically conditionally T-independent* given Z — $I_T(X, Y|Z)$ — if, for any $z \in \mathbf{Z}$ and any pair $(x, y) \in \mathbf{X} \times \mathbf{Y}$,

$$\pi_{XYZ}(x, y, z) = T\left(T\left(\pi_{X|_T Z}(x|_T z), \pi_{Y|_T Z}(y|_T z)\right), \pi_Z(z)\right).$$

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$I_T(X, Y|Z)$ satisfies so-called *semi-graphoid properties*.

Example

Joint possibility distribution

π_{XY}	$Y = 0$	$Y = 1$	
$X = 0$	1	0.7	
$X = 1$	0.5	0.3	

Example

Joint possibility distribution

π_{XY}	$Y = 0$	$Y = 1$	π_X
$X = 0$	1	0.7	1
$X = 1$	0.5	0.3	0.5
π_Y	1	0.7	

Example

Marginal possibility distributions

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π_{XY}	$Y=0$	$Y=1$	π_X
$X=0$	1	0.7	1
$X=1$	α	β	0.5
π_Y	1	0.7	

Example

Set of extensions

π_{XY}	$Y=0$	$Y=1$	π_X
$X=0$	1	0.7	1
$X=1$	α	β	0.5
π_Y	1	0.7	

$$\alpha, \beta \leq 0.5, \max(\alpha, \beta) = 0.5$$

Example

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π_{XY}	$Y=0$	$Y=1$	π_X
$X=0$	1	0.7	1
$X=1$	α	β	0.5
π_Y	1	0.7	

$$\alpha = 0.5, \beta \in [0, 0.5]$$

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Example

T-product extensions

π_{XY}	$Y = 0$	$Y = 1$	π_X
$X = 0$	1	0.7	1
$X = 1$	0.5	$T(0.7, 0.5)$	0.5
π_Y	1	0.7	

Marginal problem

We deal with joint possibility distributions $\pi(x_N)$ on

$$\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n,$$

and their marginals $\pi(x_K)$ ($K \subseteq N$) on its subspaces

$$\mathbf{X}_K = \bigtimes_{i \in K} \mathbf{X}_i.$$

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Let $\mathcal{K}$ be a system of nonempty subsets of N and

$$\mathcal{S} = \{\pi_K(x_K)\}_{K \in \mathcal{K}}$$

set of lowdimensional possibility distributions.

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Problem

Does there exist a joint possibility distribution $\pi(x_N)$ on $\mathbf{X}_N$ such that

$$\pi(x_K) = \pi_K(x_K)?$$

Operators of composition

Let T be a continuous t-norm and $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$ be two possibility distributions defined on $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Then we define:

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operator of right composition

$$\pi_1 \triangleright_T \pi_2 \left(x_{K_1 \cup K_2}\right) \quad = \quad T\left(\pi_1\left(x_{K_1}\right), \pi_2\left(x_{K_2}\right) \bigtriangleup_T \pi_2\left(x_{K_1 \cap K_2}\right)\right),$$

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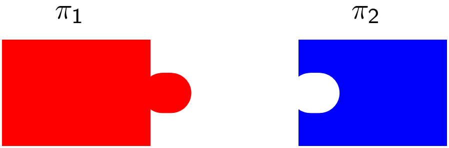
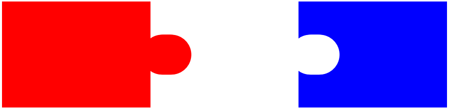
$$\pi_1 \triangleleft_T \pi_2 \left(x_{K_1 \cup K_2}\right) = T\left(\pi_1\left(x_{K_1}\right) \triangle_T \pi_1\left(x_{K_1 \cap K_2}\right), \pi_2\left(x_{K_2}\right)\right).$$

Operator of right composition

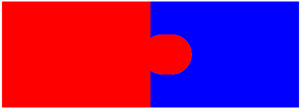


Operator of right composition

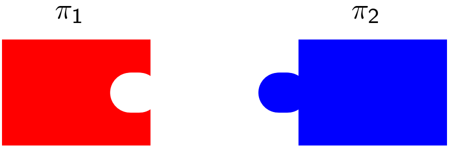
π_1 π_2



$\pi_1 \triangleright_T \pi_2$



Operator of left composition



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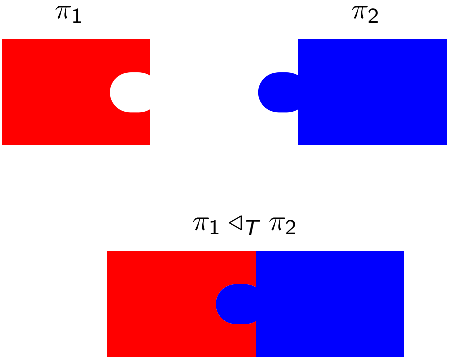


Basic properties

Lemma

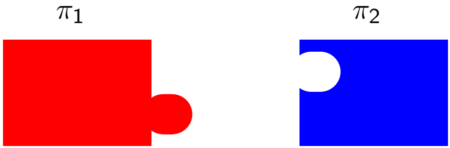
Let T be a continuous t-norm and $\pi_1(x_{K_1})$ and $\pi_2(x_{K_2})$ be two distributions on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively. Then

1. $\pi_1 \triangleright_T \pi_2$ *is a possibility distribution on* $\mathbf{X}_{K_1 \cup K_2}$,
2. $(\pi_1 \triangleright_T \pi_2)(x_{K_1}) = \pi_1(x_{K_1})$,
3. $(\pi_1 \triangleleft_T \pi_2)(x_{K_2}) = \pi_2(x_{K_2})$,
4. $(\pi_1 \triangleright_T \pi_2)(x_{K_1 \cup K_2}) = (\pi_1 \triangleleft_T \pi_2)(x_{K_1 \cup K_2})$

   for any continuous t-norm T iff π_1 and π_2 are projective, i.e.

$$\pi_1(x_{K_1 \cap K_2}) = \pi_2(x_{K_2 \cap K_1}).$$

Operator of right composition



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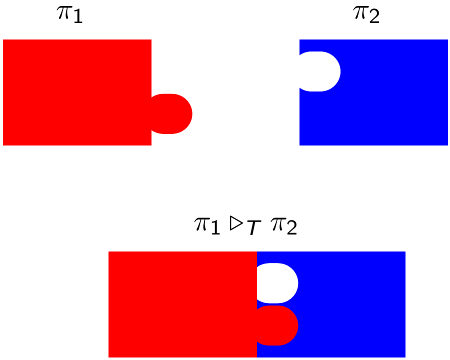

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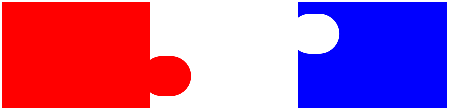
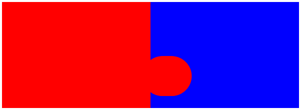


$\pi_1 \triangleright_T \pi_2$



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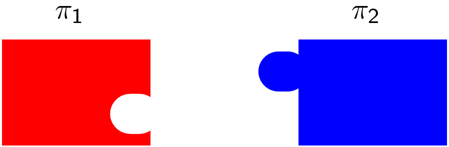

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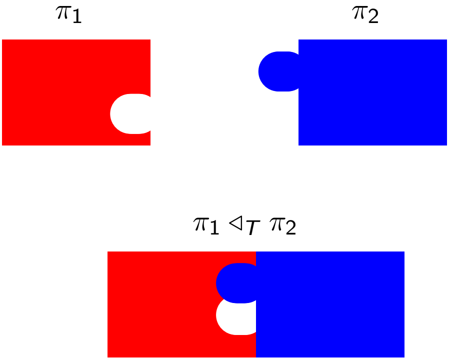

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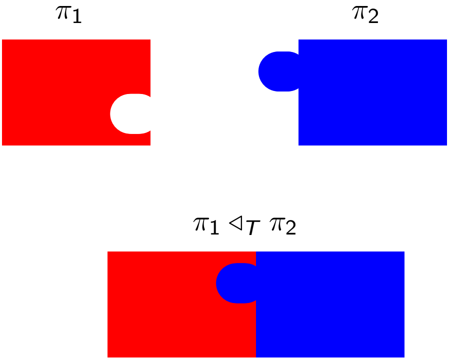

Relation to T-independence

Theorem

Let T be a continuous t-norm and π be a possibility distribution of $X_{K_1 \cup K_2}$ with marginals π_1 and π_2 of X_{K_1} and X_{K_2}, respectively. Then

$$\begin{aligned} \pi(x_{K_1 \cup K_2}) &= (\pi_1 \triangleright_T \pi_2)(x_{K_1 \cup K_2}) \\ &= (\pi_1 \triangleleft_T \pi_2)(x_{K_1 \cup K_2}), \end{aligned}$$

if and only if $X_{K_1 \setminus K_2}$ and $X_{K_2 \setminus K_1}$ are conditionally independent, given $X_{K_1 \cap K_2}$.

Generating sequences

The operator $\triangleright_T$ (as well as $\triangleleft_T$) is neither commutative nor associative. Therefore, generally

$$(\pi_1 \triangleright_T \pi_2) \triangleright_T \pi_3 \neq \pi_1 \triangleright_T (\pi_2 \triangleright_T \pi_3).$$

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Lemma

Let T be a continuous t-norm and π_1, π_2 and π_3 be defined on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}$ and $\mathbf{X}_{K_3}$, respectively, such that K_1 and K_3 are disjoint. Then

$$(\pi_1 \triangleright_T \pi_2) \triangleright_T \pi_3 = \pi_1 \triangleright_T (\pi_2 \triangleright_T \pi_3).$$

Generating sequences

Consider a sequence of possibility distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_m(x_{K_m})$ and the expression

$$\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright_T \pi_m.$$

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$$\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright_T \pi_m.$$

We always apply the operators from left to right, i.e.

$$\pi_1 \triangleright_T \pi_2 \triangleright_T \pi_3 \triangleright_T \ldots \triangleright_T \pi_m = (\ldots((\pi_1 \triangleright_T \pi_2) \triangleright_T \pi_3) \triangleright_T \ldots \triangleright_T \pi_m).$$

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It defines a multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$.

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$$\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright_T \pi_m.$$

We always apply the operators from left to right, i.e.

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It defines a multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$.Therefore, for any permutation $i_1, i_2, \ldots, i_m$ of indices $1, \ldots, m$

$$\pi_{i_1} \triangleright_T \pi_{i_2} \triangleright \ldots \triangleright_T \pi_{i_m}$$

defines also a (generally different) multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$.

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Consider a sequence of possibility distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_m(x_{K_m})$ and the expression

$$\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright_T \pi_m.$$

Generating sequences

Consider a sequence of possibility distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_m(x_{K_m})$ and the expression

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Similarly

$$\pi_1 \triangleleft_T \pi_2 \triangleleft_T \ldots \triangleleft_T \pi_m$$

defines a multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$.

Generating sequences

Consider a sequence of possibility distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_m(x_{K_m})$ and the expression

$$\pi_1 \rhd_T \pi_2 \rhd_T \ldots \rhd_T \pi_m.$$

Similarly

$$\pi_1 \lhd_T \pi_2 \lhd_T \ldots \lhd_T \pi_m$$

defines a multidimensional distribution of $X_{K_1 \cup \ldots \cup K_m}$. Nevertheless, they are very different from the computational point of view. In the first case we need to compute $|K_m \cap (K_1 \cup \ldots \cup K_{m-1})|$-dimensional marginal of

$$\pi_m(x_{K_m}),$$

while in the second case the same marginal of

$$\pi_1 \lhd_T \pi_2 \lhd_T \ldots \lhd_T \pi_{m-1}(x_{K_1 \cup \ldots \cup K_{m-1}}).$$

T-perfect sequences

An ordered sequence of possibility distributions $\pi_1, \pi_2, \ldots, \pi_m$ is said to be *T-perfect* if

$$\begin{aligned}
\pi_1 \triangleright_T \pi_2 &= \pi_1 \triangleleft_T \pi_2, \\
\pi_1 \triangleright_T \pi_2 \triangleright_T \pi_3 &= \pi_1 \triangleleft_T \pi_2 \triangleleft_T \pi_3, \\
&\vdots \\
\pi_1 \triangleright_T \cdots \triangleright_T \pi_m &= \pi_1 \triangleleft_T \cdots \triangleleft_T \pi_m.
\end{aligned}$$

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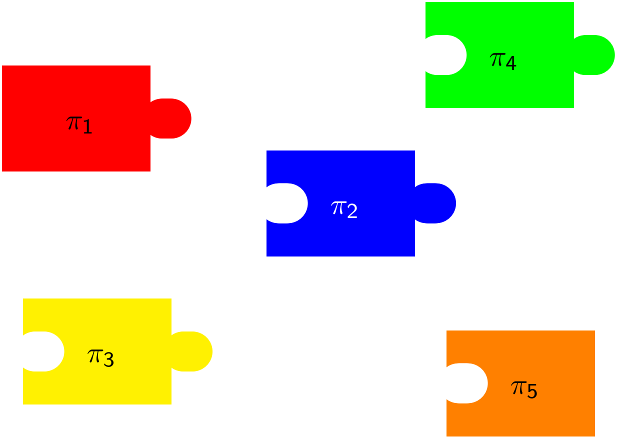
\pi_1 \triangleright_T \pi_2 \triangleright_T \pi_3 &= \pi_1 \triangleleft_T \pi_2 \triangleleft_T \pi_3, \\
&\vdots \\
\pi_1 \triangleright_T \cdots \triangleright_T \pi_m &= \pi_1 \triangleleft_T \cdots \triangleleft_T \pi_m.
\end{aligned}
$$

Theorem

The sequence $\pi_1, \pi_2, \ldots, \pi_m$ is T-perfect iff all the distributions $\pi_1, \pi_2, \ldots, \pi_m$ are marginal to distribution $\pi_1 \triangleright_T \pi_2 \triangleright_T \ldots \triangleright \pi_m$.

Perfect sequence of lowdimensional distributions



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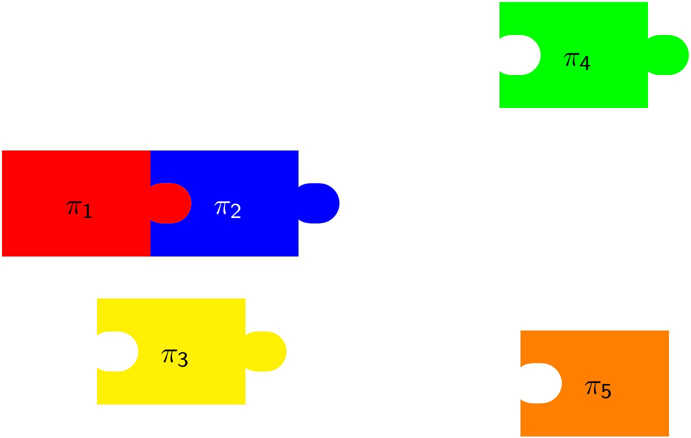

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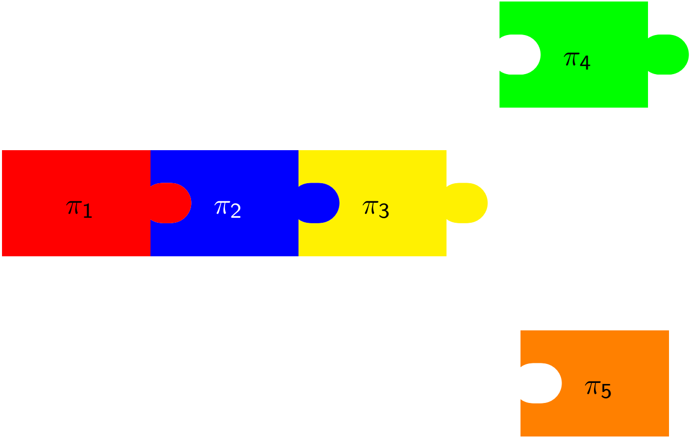

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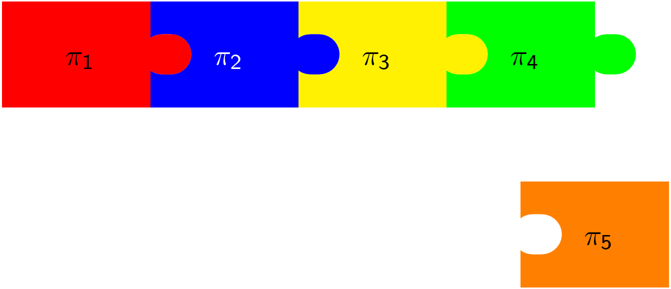

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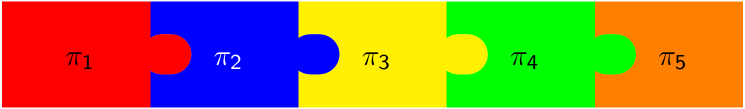
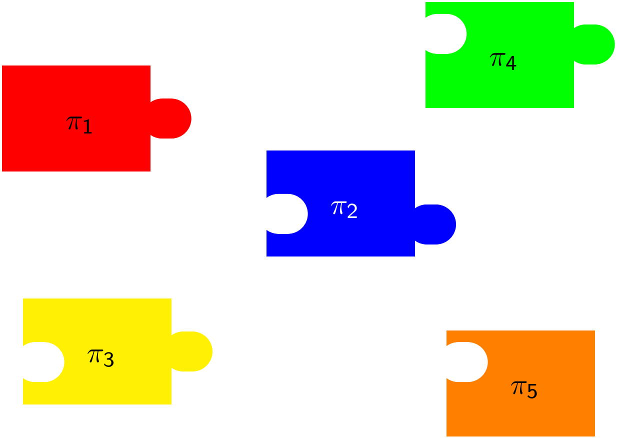
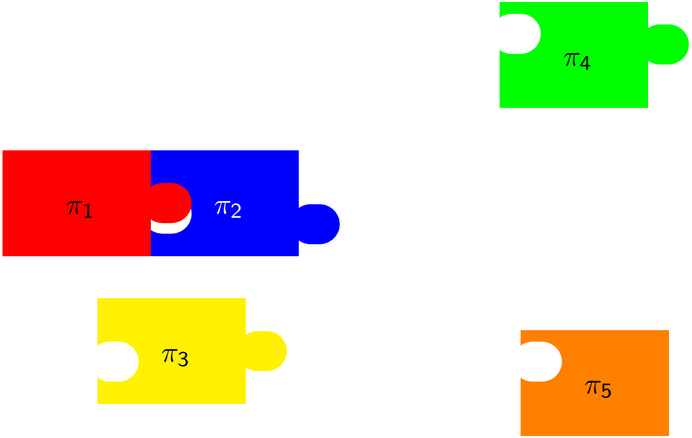


Perfect sequence

Not perfect sequence of lowdimensional distributions



Not perfect sequence of lowdimensional distributions



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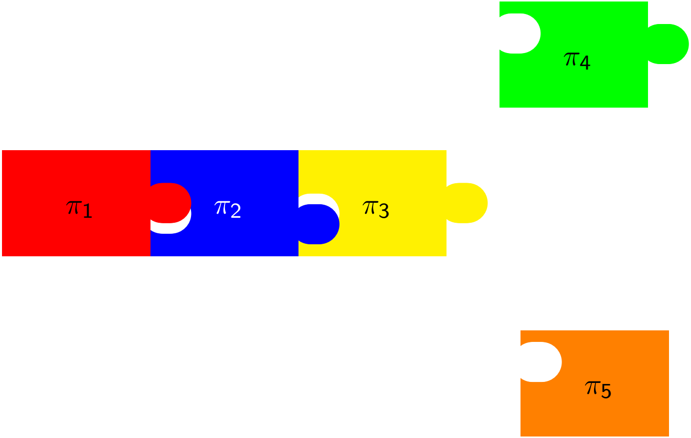

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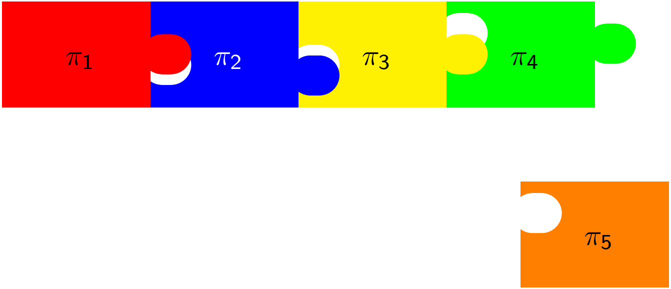

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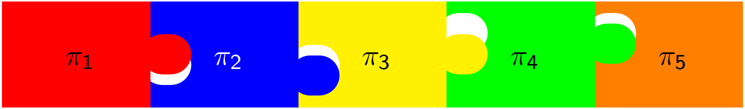

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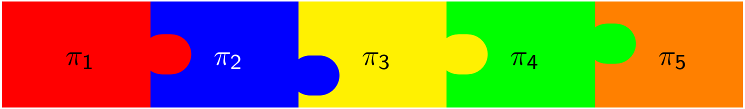

"Perfectized" sequence

Example

$\pi_1(x_1, x_3)$ X_3	0	1	2
$X_1 = 0$	1	1	1
$X_1 = 1$	.5	.7	.9

$\pi_2(x_2, x_3)$ X_3	0	1	2
$X_2 = 0$	1	1	1
$X_2 = 1$	.5	.4	.3

Example

X_1	X_2	X_3	$\pi_1 \triangleright_T \pi_2(X_1, X_2, X_3)$		
			G	Π	L
0	0	0	1	1	1
0	0	1	1	1	1
0	0	2	1	1	1
0	1	0	.5	.5	.5
0	1	1	.4	.4	.4
0	1	2	.3	.3	.3
1	0	0	.5	.5	.5
1	0	1	.7	.7	.7
1	0	2	.9	.9	.9
1	1	0	.5	.25	0
1	1	1	.4	.28	.1
1	1	2	.3	.27	.2

Example

X_3	$\pi_1 \triangleright_T \pi_2(X_3 \mid X_1 = 1, X_2 = 1)$		
	G	Π	L
0	1	25/28	.8
1	.4	1	.9
2	.3	27/28	1

Upper envelopes of sets of probability distributions

With any possibility distribution π on $\mathbf{X}$ we can associate a class of probability distributions $\mathcal{M}(\pi)$ on $\mathbf{X}$ dominated by it, i.e.,

$$\mathcal{M}(\pi) = \{p : p(x) \leq \pi(x) \quad \forall x \in \mathbf{X}\}.$$

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Theorem

Let $\pi_1, \pi_2, \ldots, \pi_m$ be a min-perfect sequence of possibility distributions and $\mathcal{M}(\pi_1), \mathcal{M}(\pi_2), \ldots, \mathcal{M}(\pi_m)$ corresponding sets of probability distributions. Then

$$\pi_1 \rhd_G \pi_2 \rhd_G \cdots \rhd_G \pi_m$$

is the upper envelope of the set of all extensions of projective probability distributions from $\mathcal{M}(\pi_1), \mathcal{M}(\pi_2), \ldots, \mathcal{M}(\pi_m)$.

Upper envelopes of sets of probability distributions

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$$\pi_1 \triangleright_{\Pi} \pi_2 \triangleright_{\Pi} \cdots \triangleright_{\Pi} \pi_m$$

is an upper envelope of the probability distributions

$$p_1 \triangleright p_2 \triangleright \cdots \triangleright p_m,$$

where $p_1, p_2, \ldots p_m$ form perfect sequences of probability distributions from $\mathcal{M}(\pi_1), \mathcal{M}(\pi_2), \ldots, \mathcal{M}(\pi_m)$.

Possibilistic trees

Possibilistic trees (de Campos and Huete, FSS 1999) are based on the following simple idea. If $I_T(X, Y|Z)$, then the joint distribution $\pi(x, y, z)$ of X, Y, Z can be obtained from its marginals $\pi(x, z)$ and $\pi(y, z)$.

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Let us assume variables $X_1, \ldots, X_n$ such that $I_T(\{X_j\}_{j<i}\{X_j\}_{j>i}|i)$, then the joint possibility distribution of these variables can be obtained form the marginals $\pi(x_1, \ldots, x_i)$ and $\pi(x_i, \ldots, x_n)$.

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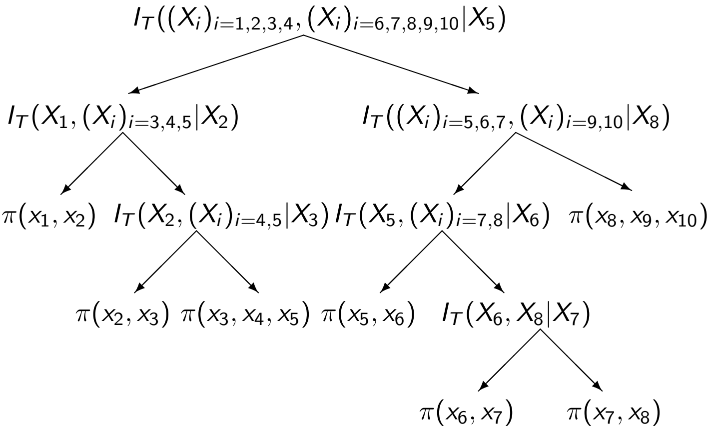
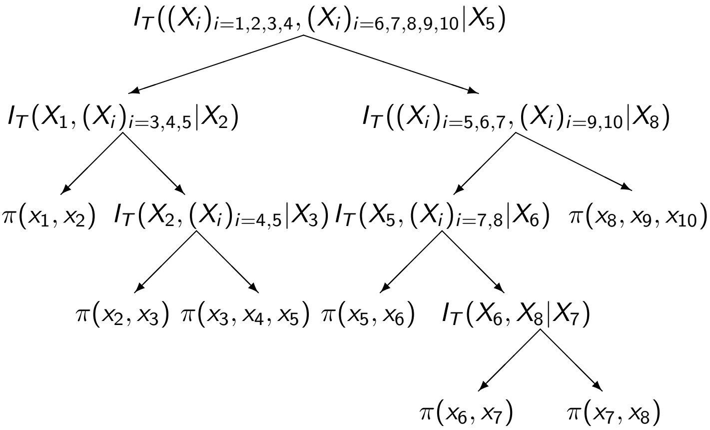
Resulting possibilistic tree $\mathcal{T}$ consists of two kinds of nodes — *leaf nodes* (which store marginal possibility distributions) and *internal nodes* (storing conditional independence statements).

Example

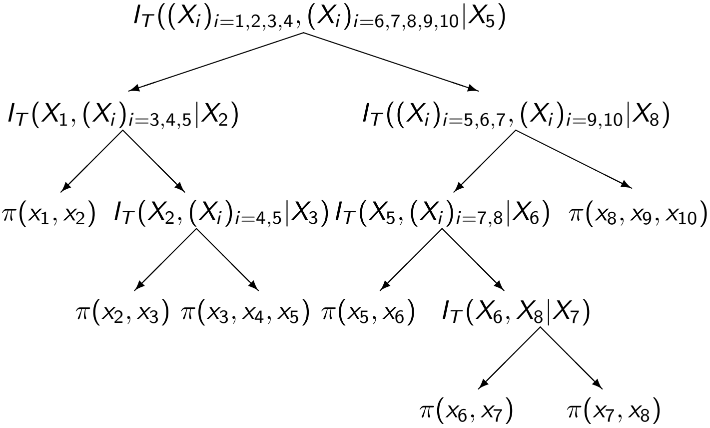

$I_T((X_i)_{i=1,2,3,4},(X_i)_{i=6,7,8,9,10}|X_5)$
$I_T(X_1,(X_i)_{i=3,4,5}|X_2)$
$I_T((X_i)_{i=5,6,7},(X_i)_{i=9,10}|X_8)$
$\pi(x_1,x_2)$
$I_T(X_2,(X_i)_{i=4,5}|X_3)$
$I_T(X_5,(X_i)_{i=7,8}|X_6)$
$\pi(x_8,x_9,x_{10})$
$\pi(x_2,x_3)$
$\pi(x_3,x_4,x_5)$
$\pi(x_5,x_6)$
$I_T(X_6,X_8|X_7)$
$\pi(x_6,x_7)$
$\pi(x_7,x_8)$

Example



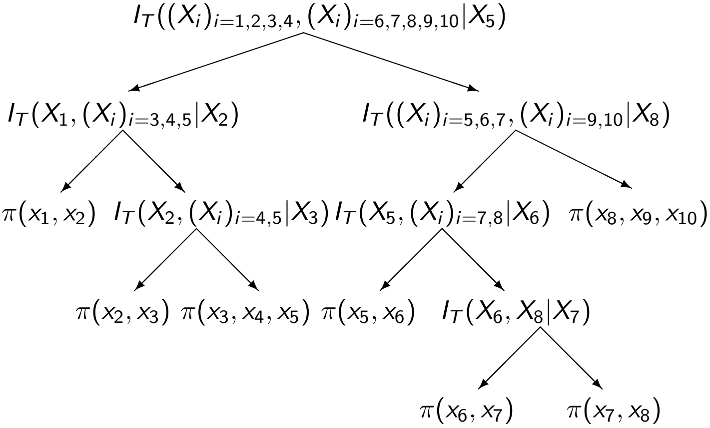
$$\pi(x_2, x_3) \triangleright_T \pi(x_3, x_4, x_5)$$

Example



$$\pi(x_1, x_2) \triangleright_T (\pi(x_2, x_3) \triangleright_T \pi(x_3, x_4, x_5))$$

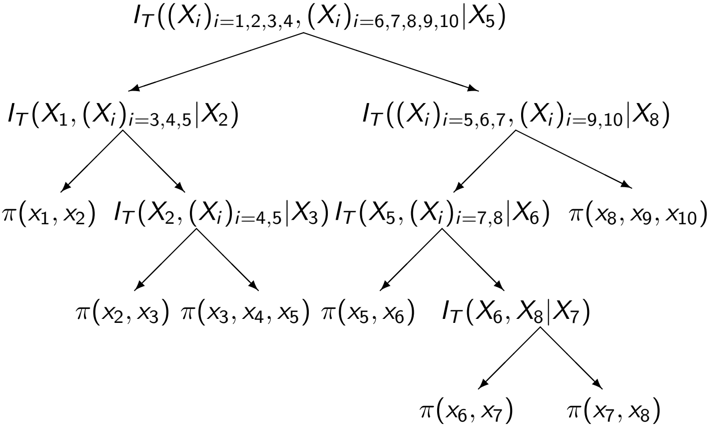
Example



$$\pi(x_1,x_2) \triangleright_T (\pi(x_2,x_3) \triangleright_T \pi(x_3,x_4,x_5))$$

$$\pi(x_6,x_7) \triangleright_T \pi(x_7,x_8)$$

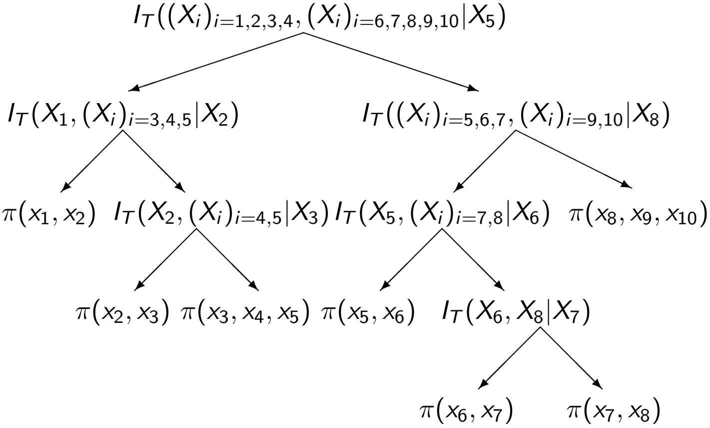
Example



$$\pi(x_1,x_2) \triangleright_T (\pi(x_2,x_3) \triangleright_T \pi(x_3,x_4,x_5))$$

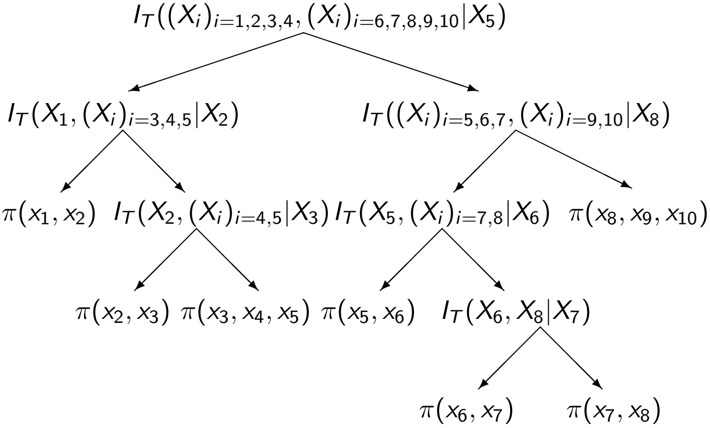
$$\pi(x_5,x_6) \triangleright_T (\pi(x_6,x_7) \triangleright_T \pi(x_7,x_8))$$

Example



$$\pi(x_1, x_2) \triangleright_T (\pi(x_2, x_3) \triangleright_T \pi(x_3, x_4, x_5))$$
$$(\pi(x_5, x_6) \triangleright_T (\pi(x_6, x_7) \triangleright_T \pi(x_7, x_8))) \triangleright_T \pi(x_8, x_9, x_{10})$$

Example



$$(\pi(x_1,x_2) \triangleright_T (\pi(x_2,x_3) \triangleright_T \pi(x_3,x_4,x_5))) \triangleright_T ((\pi(x_5,x_6) \triangleright_T (\pi(x_6,x_7) \triangleright_T \pi(x_7,x_8)))) \triangleright_T \pi(x_8,x_9,x_{10}))$$

Example — continued

$$(\pi(x_1, x_2) \triangleright_T (\pi(x_2, x_3) \triangleright_T \pi(x_3, x_4, x_5))) \\ \triangleright_T ((\pi(x_5, x_6) \triangleright_T (\pi(x_6, x_7) \triangleright_T \pi(x_7, x_8))) \triangleright_T \pi(x_8, x_9, x_{10}))$$

Example — continued

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Example — continued

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Dependence trees

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For each dependence tree one can construct a perfect sequence $\pi_1, \ldots, \pi_m$ of distributions of variables $X_{K_1}, X_{K_2}, \ldots, X_{K_m}$, respectively. These distributions are such that each $\{X_i\}_{i \in K_k}$ equals some $cl(X_j) = \{X_j\} \cup pa(X_j)$ and $\pi_1 \triangleright \ldots \triangleright \pi_m$ equals the distribution represented by the dependence tree.

Directed possibilistic graphs

Directed possibilistic graph (or *possibilistic belief network*) is a possibilistic counterpart of Bayesian network:

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For each directed possibilistic graph one can construct a perfect sequence $\pi_1, \ldots, \pi_m$ of distributions of variables $X_{K_1}, X_{K_2}, \ldots, X_{K_m}$, respectively. These distributions are such that each $\{X_i\}_{i \in K_k}$ equals some $cl(X_j) = \{X_j\} \cup pa(X_j)$ and $\pi_1 \triangleright \ldots \triangleright \pi_m$ equals the distribution represented by the directed possibilistic graph.

Directed possibilistic graphs

Having a perfect sequence $\pi_1, \pi_2, \ldots, \pi_m$ (π_k being the distribution of X_{K_k}), we first order (in an arbitrary way) all the variables for which at least one of the distributions π_k is defined, i.e.

$$\{X_1, X_2, X_3, \ldots, X_n\} = \{X_i\}_{i \in K_1 \cup \ldots \cup K_m}.$$

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1. the nodes are all the variables $X_1, X_2, X_3, \ldots, X_n$;
2. there is an edge $(X_i \rightarrow X_j)$ if there exists a distribution π_k such that both $i, j \in K_k$, $j \notin K_1 \cup \ldots \cup K_{k-1}$ and either $i \in K_1 \cup \ldots \cup K_{k-1}$ or $i < j$.

Example

$$\pi_1(G,B), \pi_2(T), \pi_3(B), \pi_4(D,T,G), \pi_5(R,B), \pi_6(W,R,D)$$

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$$G, B, T, D, R, W$$

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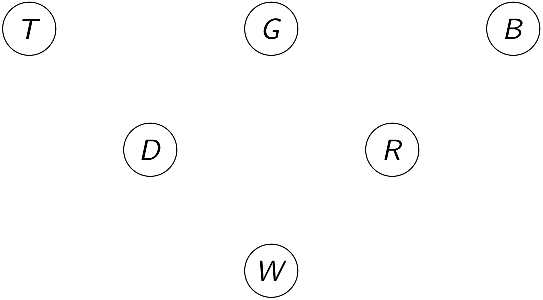

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$$\pi_1(G, B), \pi_2(T), \pi_3(B), \pi_4(D, T, G), \pi_5(R, B), \pi_6(W, R, D)$$

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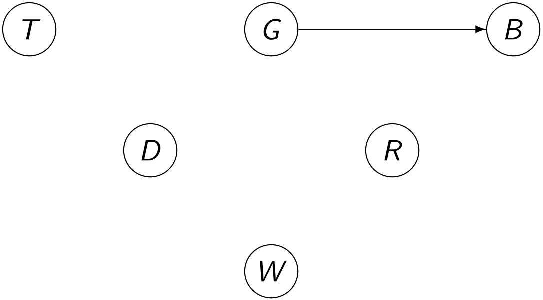

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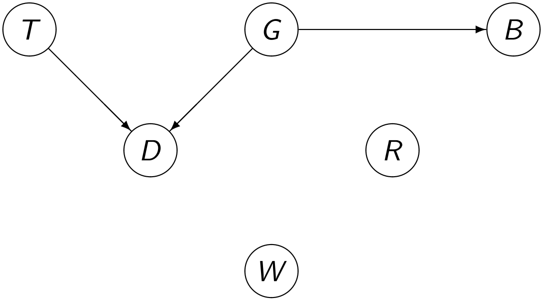

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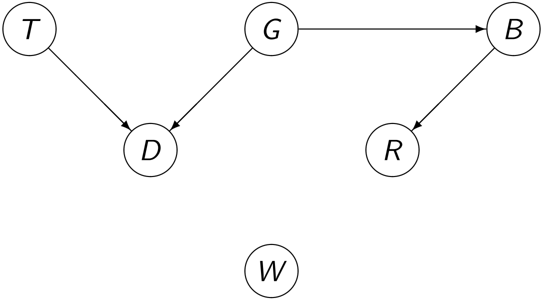

Example

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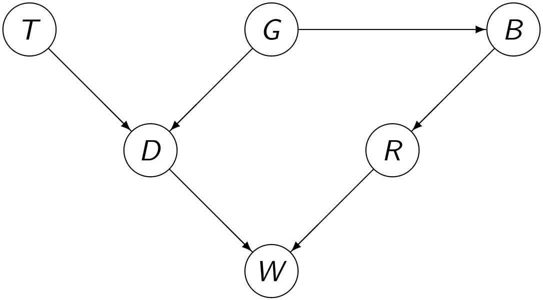
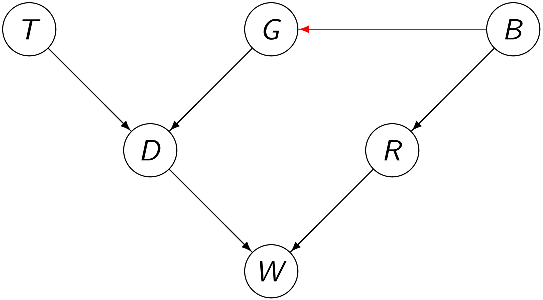
$$G, B, T, D, R, W$$



Example

$$\pi_1(G, B), \pi_2(T), \pi_3(B), \pi_4(D, T, G), \pi_5(R, B), \pi_6(W, R, D)$$

$$B, G, T, D, R, W$$



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- There exist nice probabilistic interpretation of compositional models based on Gödel's and product t-norms.
- Three types of graphical possibilistic models can be expressed by a perfect sequence of low-dimensional distributions.
- There exists a procedure by which any perfect sequence of low-dimensional distributions can be transformed into a directed possibilistic graph (or a possibilistic belief network).

THANK YOU FOR YOUR ATTENTION.