L-valued bornologies generated by fuzzy metrics

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Bornology. Bornological space

Bornology on a set X

A family $\mathfrak{B} \subseteq 2^X$ is called a bornology on a set X if

1. $\bigcup\{B | B \in \mathfrak{B}\} = X$;
2. If $B \in \mathfrak{B}$ and $C \subset B$ then $C \in \mathfrak{B}$;
3. If $B_1, B_2 \in \mathfrak{B}$ then $B_1 \bigcup B_2 \in \mathfrak{B}$.

Bornological space

A pair $(X, \mathfrak{B})$ is called a bornological space.

Bounded mapping

Given two bornological spaces $(X, \mathfrak{B}_X)$, $(Y, \mathfrak{B}_Y)$ a mapping $f : X \to Y$ is called bounded if $A \in \mathfrak{B}_X \Longrightarrow f(A) \in \mathfrak{B}_Y$.

Bornology

In a certain sense from the analitic point of view a bornological space can be viewed as a counterpart of topological space if one is mainly interested in the property of boundedness of mappings and not in their property of continuity.

The aim of our research is to develop the basics of the theory of bornological structures in the context of fuzzy sets.

- Crisp bornological structures on L-powersets (Basics are developed in the works by M.Abel, A.Šostak, I. Uļjane)
- Fuzzy bornological structures on powersets (Basics to be considered in the present talk)
- M-valued bornological structures on L-powersets (To be developed in the perspective)

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L-valued bornological spaces

L-valued bornological spaces

The context of work

cl-monoid

A cl-monoid is a tuple $(L, \leq, \wedge, \vee, *)$ where

1) $(L, \leq, \wedge, \vee)$ is a complete infinitely distributive lattice with bottom 0_L and top 1_L elements
2) $* : L \times L \to L$ is a binary associative operation;
3) $*$ distributes over arbitrary joins: $\alpha * \left(\bigvee_i \beta_i\right) = \bigvee_i (\alpha * \beta_i)$.

De Morgan algebras

A De Morgan algebra is a tuple $(L, \leq, \wedge, \vee, ^c)$ where

1) $(L, \leq, \wedge, \vee)$ is a complete infinitely distributive lattice with bottom 0_L and top 1_L elements
2) $^c : L \to L$ is an order reversing involution;
3) De Morgan law is fulfilled: $(\alpha \vee \beta)^c = \alpha^c \wedge \beta^c$; $\quad (\alpha \wedge \beta)^c = \alpha^c \vee \beta^c$.

L-valued bornological space

L-valued bornology

An L-valued bornology on a set X is a mapping $\mathcal{B} : \mathbf{2}^X \to L$, such that

1) $\forall\, x \in X \quad \mathcal{B}(\{x\}) = 1$;
2) If $U \subset V \subset X$ then $\mathcal{B}(V) \leq \mathcal{B}(U)$;
3) $\forall\, U, V \subset X \quad \mathcal{B}(U \cup V) \geq \mathcal{B}(U) * \mathcal{B}(V)$.

L-valued bornological space

The pair $(X, \mathcal{B})$ will be called an L-valued bornological space.

L-valued bornological space

Remark

If A is a finite subset of a set X then $\mathcal{B}(A) = 1$.

Remark

If $* = \wedge$, the last axiom is

(3') $\forall\, U, V \subset X \quad \mathcal{B}(U \cup V) = \mathcal{B}(U) \wedge \mathcal{B}(V)$,

and therefore the axiom (2) can be omitted.

On the family $\mathfrak{B}(2^X)$ of all L-valued bornologies on set X, we introduce an partial order by setting

$$\mathcal{B}_1 \preceq \mathcal{B}_2 \text{ iff } \quad \forall\, A \subset X \quad \mathcal{B}_2(A) \leq \mathcal{B}_1(A).$$

Theorem

$\mathfrak{B}(2^X)$ is a complete lattice.

Top and botom L-valued bornologies in $(\mathfrak{B}(2^L), \preceq)$

The bottom L-valued bornology

$$\mathcal{B}_{\perp}(A) = 1 \quad \forall\, A \subset X$$

The top L-valued bornology

$$\mathcal{B}_{\top}(A) = \begin{cases} 1, & \text{if } |A| < \aleph_0, \\ 0, & \text{otherwise}. \end{cases}$$

Construction of a supremum and an infimum in $(\mathfrak{B}(2^X), \preceq)$

Let $\mathfrak{B}_I \subset \mathfrak{B}(2^X)$ and $\mathfrak{B}_I = \{\mathcal{B}_i | \quad i \in I\}$

$$\forall A \subset X \quad \left(\bigvee \mathfrak{B}_I\right)(A) = \inf_{i \in I}\{\mathcal{B}_i(A)\} \in \mathfrak{B}(2^X).$$

From existence of supremum $\bigvee \mathfrak{B}_I$

follows existence of infimum $\bigwedge \mathfrak{B}_I$.

Bounded mappings between L-valued bornological spaces

Bounded mappings

By a bounded mapping from an L-valued bornological space $(X, \mathcal{B}_X)$ to an L-valued bornological space $(Y, \mathcal{B}_Y)$ we call

$$f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y) \text{ such that}$$

$$\mathcal{B}_Y(f(A)) \geq \mathcal{B}_X(A) \quad \forall A \subset X.$$

Theorem

L-valued bornological spaces as objects and bounded mappings between them as morphisms form a categoty. This category is denoted *L-BORN*.

Level bornologies of an L-valued bornology

Level bornologies

Let $\mathcal{B} : \mathbf{2}^X \to L$ be an L-bornology on a set X, $\lambda \in L$

$$\mathcal{B}_\lambda = \{A \in 2^X \mid \mathcal{B}(A) \geq \lambda\}$$

called λ-level bornology of the L-valued bornology $\mathcal{B}$.

Remark

In case when λ is an idempotent element in the *cl*-monoid $(L, \leq, \vee, \wedge, *)$ family $\mathcal{B}_\lambda$ is a bornology on a set X.

Example

$$\mathcal{B} : 2^X \to [0,1], \text{ such that}$$

$$\forall\, A \subset X \quad \mathcal{B}(A) = \begin{cases} 1, & \textit{if } |A| < \aleph_0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

$$\mathcal{B}_{\frac{2}{3}} = \{A | A \subset X \text{ and } |A| < \aleph_0\}$$

$$\mathcal{B}_{\frac{1}{5}} = 2^X$$

Construction of an L-valued bornology

Let $\mathcal{C} = \{C_\alpha \mid \alpha \in L\}$ be a family of bornologies on a set X s.t.

$$\alpha \leq \beta \Longrightarrow C_\alpha \subseteq C_\beta.$$

For a given $A \in 2^X$ let

$$\mathcal{B}(A) = \bigvee \{\alpha^c \mid A \in C_\alpha\}.$$

Theorem

The mapping

$$\mathcal{B} : 2^X \to L$$

constructed above is an L-valued bornology on X. Besides if $\mathcal{C}$ is uppersemicontinuous from above, then

$$\mathcal{B}_\lambda = C_{\lambda^c}.$$

Fuzzy Metric (A.Gegore, P.Veeramani 1994)

Definition of fuzzy metric space

A fuzzy metric on a set X is a pair $(M, *)$ such that M is a fuzzy set on $X \times X \times [0, \infty)$ and * is a continuous t-norm satisfying the following conditions:

(1FM) $M(x, y, t) > 0$ for all $x, y \in X$, for all $t > 0$;

(2FM) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(3FM) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$, for all $t > 0$;

(4FM) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \quad \forall x, y, z \in X \quad \forall t, s > 0$;

(5FM) $M(x, y, \cdot)$ is continuous for each $x, y \in X$.

The triple $(X, M, *)$ is called a fuzzy metric spaces.

- A.George, P.Veeramani, On some results of fuzzy metric space, Fuzzy Sets and Systems 64(1994) 395-399.

Strong Fuzzy Metric

Strong Fuzzy Metric

A fuzzy metric $M : X \times X \times [0; \infty) \to (0; 1]$ is called strong if it satisfies, in addition to the properties $(1\mathrm{FM})$ - $(5\mathrm{FM})$, the following stronger version of the axiom $(4\mathrm{FM})$

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t) \quad \forall x, y, z \in X, \quad \forall t > 0$$

Examples of fuzzy metrics

Example 1.

Let $f : X \to R^+$ be a one-to-one function and $g : R^+ \to [0; \infty)$ be an incrreasing continuous function. Fixed $\alpha, \beta > 0$, define M by

$$M(x, y, t) = \left(\frac{(min\{f(x), f(y)\})^\alpha + g(t)}{(max\{f(x), f(y)\})^\alpha + g(t)} \right)^\beta .$$

Then, $(M, \cdot)$ is fuzzy metric on X.

- V.Gregori, S.Morillas, A. Sepena, Examples of fuzzy metrics and aplications, Fuzzy Sets and Systems 170 (2011) 95-111

Examples of fuzzy metrics

Example 1.1.

Let $X = R^+$, and let g be the identity function. Then (1) becomes

$$M(x, y, t) = \frac{min\{x, y\} + t}{max\{x, y\} + t}$$

Then, $(M, \cdot)$ is strong fuzzy metric on X.

- P.Veeramani, Best approximation in fuzzy metric spaces, Journal of Fuzzy Mathematics 9 (2001) 75-80.

Examples of fuzzy metrics

Example 2.

Let (X, d) be a bounded metric space and suppose $d(x, y) < k$ for all $x, y \in X$. Let $g : R^+ \to (k; +\infty)$ be an increasing continuous function. Define the function M by

$$M(x, y, t) = 1 - \frac{d(x, y)}{g(t)}.$$

Then $(M, Ł)$ is a fuzzy metric on X.

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Examples of fuzzy metrics

Example 2.2.

If we take g as a constant function $g(t) = K > k$, then (2) becomes

$$M(x, y) = 1 - \frac{d(x, y)}{K}$$

and so $(M, Ł)$ is a strong is a strong fuzzy mertic.

- V.Gregori, S.Morillas, A. Sepena, Examples of fuzzy metrics and aplications, Fuzzy Sets and Systems 170 (2011) 95-111

Examples of fuzzy metrics

Example 3.

Let $\varphi : R^{+} \rightarrow [0;1)$be an increaseang continuous function. Define the function M by

$$M(x,y,t) = \begin{cases} 1, & \text{if } x = y \\ \varphi(t), & \text{otherwise.} \end{cases}$$

and so $(M, \wedge)$ is a fuzzy mertic on X.

- V.Gregori, S.Morillas, A. Sepena, Examples of fuzzy metrics and aplications, Fuzzy Sets and Systems 170 (2011) 95-111

Given a fuzzy metric space $(X, M, *)$ a (crisp) topology can be introduce on X as follows: Let $(X, M, *)$ be a fuzzy metric space.
Following

- V.Gregori, S.Morillas, A. Sepena, On a class of completable fuzzy metrics, Fuzzy Sets and Systems 161 (2010) 2193-2205

given a point $x \in X$, a positive real number ε and a non-negative real number t we define a ball at the level t with center x and radius ε as the set

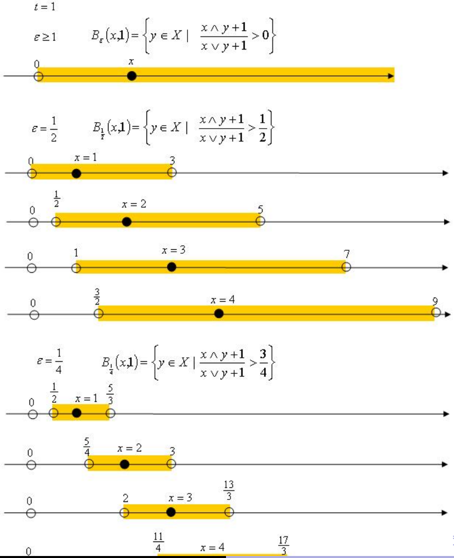
$$B_\varepsilon(x, t) = \{y \in X \mid M(x, y, t) > 1 - \varepsilon\}.$$

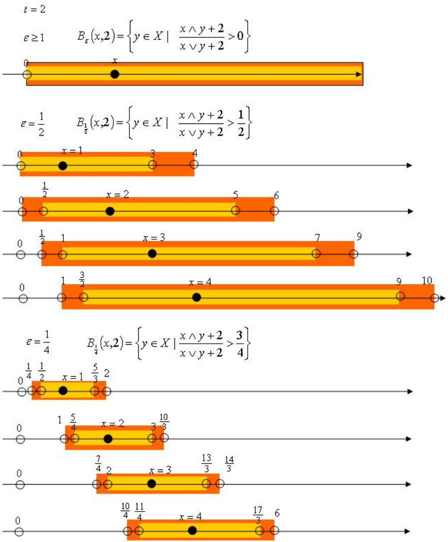
$$t \leq s \Longrightarrow B_\varepsilon(x,t) \subseteq B_\varepsilon(x,s)$$

$$\varepsilon \leq \delta \Longrightarrow B_\varepsilon(x,t) \subseteq B_\delta(x,t)$$

Let $\quad M(x,y,t) = \dfrac{x \wedge y + t}{x \vee y + t} \qquad x, y \in R+$ and $t > 0$

$$B_\varepsilon(x,t) = \left\{ y \in X \mid \frac{x \wedge y + t}{x \vee y + t} > 1 - \varepsilon \right\}.$$


$t = 1$
$\varepsilon \geq 1$
$B_\varepsilon(x,1) = \left\{ y \in X \mid \frac{x \wedge y + 1}{x \vee y + 1} > 0 \right\}$
0
x
$\varepsilon = \frac{1}{2}$
$B_{\frac{1}{2}}(x,1) = \left\{ y \in X \mid \frac{x \wedge y + 1}{x \vee y + 1} > \frac{1}{2} \right\}$
0
$x = 1$
3
0
$\frac{1}{2}$
$x = 2$
5
0
1
$x = 3$
7
0
$\frac{3}{2}$
$x = 4$
9
$\varepsilon = \frac{1}{4}$
$B_{\frac{1}{4}}(x,1) = \left\{ y \in X \mid \frac{x \wedge y + 1}{x \vee y + 1} > \frac{3}{4} \right\}$
0
$\frac{1}{2}$
$x = 1$
$\frac{5}{3}$
0
$\frac{5}{4}$
$x = 2$
3
0
2
$x = 3$
$\frac{13}{3}$
0
$\frac{11}{4}$
$x = 4$
$\frac{17}{3}$


$t = 2$
$\varepsilon \geq 1$
$B_\varepsilon(x,2) = \left\{ y \in X \mid \frac{x \wedge y + 2}{x \vee y + 2} > 0 \right\}$
$\varepsilon = \frac{1}{2}$
$B_{\frac{1}{2}}(x,2) = \left\{ y \in X \mid \frac{x \wedge y + 2}{x \vee y + 2} > \frac{1}{2} \right\}$
$x = 1$
$x = 2$
$x = 3$
$x = 4$
$\varepsilon = \frac{1}{4}$
$B_{\frac{1}{4}}(x,2) = \left\{ y \in X \mid \frac{x \wedge y + 2}{x \vee y + 2} > \frac{3}{4} \right\}$
$x = 1$
$x = 2$
$x = 3$
$x = 4$

As different from the situation with topological structure in fuzzy metric space, the corresponding bornological structure of this space is essentially L-valued bornology on the powerset 2^X. To construct an L-valued (where $L = [0,1]$) bornology on the set X induced by fuzzy metric M we first fix a strongly decreasing bijection

$$\varphi : [0, \infty) \to (0, 1]$$

(as a typical example here one can take the hyperbola $\varphi(t) = \frac{1}{1+t} \forall t \in [0, \infty)$.)
Further we see natural approach how an L-valued bornology on the space $(X, M, *)$ could be defined:

L-valued bornologies induced by fuzzy metrics

Definition

Given a fuzzy metric space $(X, M, *)$ and a number $\alpha \in (0, 1]$ we call a set $A \subseteq X$ α-bounded if there exists $\varepsilon > 0$ and a point $x \in X$ such that $A \subseteq B_\varepsilon(x, \varphi^{-1}(\alpha))$.

Given a fuzzy metric space $(X, M, *)$ let $\mathcal{C}_\alpha$ stand for the family of all finite unions of α-bounded subsets of X. One can easily verify that $\mathcal{C}_\alpha$ is a crisp bornology on the set X. Besides the family $\{\mathcal{C}_\alpha \,|\alpha \in [0, 1)\}$ is nondecreasing:

$$\alpha \leq \beta \Longrightarrow \mathcal{C}_\alpha \subseteq \mathcal{C}_\beta.$$

Hence we can apply the construction which we have developed earlier in order to define an L -valued bornology $\mathcal{B} : 2^X \to I$ from the family $\{\mathcal{C}_\alpha \mid \alpha \in (0;1]\}$ of crips bornologies:

$$\mathcal{B}(A) = \bigvee\{\alpha^c \mid A \in \mathcal{C}_\alpha\},$$

where the involution $^c : [0;1] \to [0;1]$ is defined in a natural way: $\alpha^c = 1 - \alpha$.

Thank you for attention!