L-valued bornologies generated by fuzzy metrics

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11th International Conference on Fuzzy Set Theory and Applications
January 30 - February 3, 2012, Liptovsky Jan, the Slovak Republic
Introduction: Bornology on a set

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  - Boundedness in a topological vector space
  - Boundedness in a topological space
  - S.-T. Hu studied the problem of possibility to define the concept of boundedness in a topological space.
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Bornology. Bornological space

**Bornology on a set $X$**

A family $\mathcal{B} \subseteq 2^X$ is called a bornology on a set $X$ if

1. $\bigcup \{B | B \in \mathcal{B} \} = X$;
2. If $B \in \mathcal{B}$ and $C \subset B$ then $C \in \mathcal{B}$;
3. If $B_1, B_2 \in \mathcal{B}$ then $B_1 \cup B_2 \in \mathcal{B}$.

**Bornological space**

A pair $(X, \mathcal{B})$ is called a bornological space.

**Bounded mapping**

Given two bornological spaces $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ a mapping $f : X \to Y$ is called bounded if $A \in \mathcal{B}_X \implies f(A) \in \mathcal{B}_Y$.
In a certain sense from the analytic point of view a bornological space can be viewed as a counterpart of topological space if one is mainly interested in the property of boundedness of mappings and not in their property of continuity.

The aim of our research is to develop the basics of the theory of bornological structures in the context of fuzzy sets.
Crisp bornological structures on $L$-powersets (Basics are developed in the works by M. Abel, A. Šostak, I. Uljane)

Fuzzy bornological structures on powersets (Basics to be considered in the present talk)

$M$-valued bornological structures on $L$-powersets (To be developed in the perspective)
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$L$-valued bornological spaces

$L$-valued bornological spaces

$L$-valued bornologies generated by fuzzy metrics
**cl-monoid**

A cl-monoid is a tuple \((L, \leq, \land, \lor, \ast)\) where

1) \((L, \leq, \land, \lor)\) is a complete infinitely distributive lattice with bottom \(0_L\) and top \(1_L\) elements

2) \(\ast : L \times L \to L\) is a binary associative operation;

3) \(\ast\) distributes over arbitrary joins: \(\alpha \ast (\bigvee_i \beta_i) = \bigvee_i (\alpha \ast \beta_i)\).

**De Morgan algebras**

A De Morgan algebra is a tuple \((L, \leq, \land, \lor, c)\) where

1) \((L, \leq, \land, \lor)\) is a complete infinitely distributive lattice with bottom \(0_L\) and top \(1_L\) elements

2) \(c : L \to L\) is an order reversing involution;

3) De Morgan law is fulfilled: \((\alpha \lor \beta)^c = \alpha^c \land \beta^c; \quad (\alpha \land \beta)^c = \alpha^c \lor \beta^c\).
**L-valued bornology**

An *L*-valued bornology on a set *X* is a mapping $\mathcal{B} : 2^X \rightarrow L$, such that

1) $\forall x \in X \quad \mathcal{B}(\{x\}) = 1$;
2) If $U \subset V \subset X$ then $\mathcal{B}(V) \leq \mathcal{B}(U)$;
3) $\forall U, V \subset X \quad \mathcal{B}(U \cup V) \geq \mathcal{B}(U) \ast \mathcal{B}(V)$.

**L-valued bornological space**

The pair $(X, \mathcal{B})$ will be called an *L*-valued bornological space.
### Remark

If $A$ is a finite subset of a set $X$ then $B(A) = 1$.

### Remark

If $\ast = \land$, the last axiom is

$$(3') \quad \forall U, V \subset X \quad B(U \cup V) = B(U) \land B(V),$$

and therefore the axiom (2) can be omitted.
On the family $\mathcal{B}(2^X)$ of all $L$-valued bornologies on set $X$, we introduce an partial order by setting

$$\mathcal{B}_1 \preceq \mathcal{B}_2 \text{ iff } \forall A \subseteq X \quad \mathcal{B}_2(A) \leq \mathcal{B}_1(A).$$

**Theorem**

$\mathcal{B}(2^X)$ is a complete lattice.
Top and bottom $L$-valued bornologies in $(\mathcal{B}(2^L), \preceq)$

The bottom $L$-valued bornology

$$\mathcal{B}_\bot(A) = 1 \quad \forall A \subset X$$

The top $L$-valued bornology

$$\mathcal{B}_\top(A) = \begin{cases} 1, & \text{if } |A| < \aleph_0, \\ 0, & \text{otherwise} \end{cases}$$
Construction of a supremum and an infimum in 
\((\mathcal{B}(2^X), \preceq)\)

Let \(\mathcal{B}_I \subset \mathcal{B}(2^X)\) and \(\mathcal{B}_I = \{B_i \mid i \in I\}\)

\[\forall A \subset X \quad \left(\bigvee_{i \in I} \mathcal{B}_i\right)(A) = \inf_{i \in I} \{B_i(A)\} \in \mathcal{B}(2^X).\]

From existence of supremum \(\bigvee \mathcal{B}_I\)

follows existence of infimum \(\bigwedge \mathcal{B}_I.\)
Bounded mappings between $L$-valued bornological spaces

**Bounded mappings**

By a bounded mapping from an $L$-valued bornological space $(X, B_X)$ to an $L$-valued bornological space $(Y, B_Y)$ we call

$$f : (X, B_X) \rightarrow (Y, B_Y) \text{ such that}$$

$$B_Y(f(A)) \geq B_X(A) \quad \forall A \subset X.$$ 

**Theorem**

$L$-valued bornological spaces as objects and bounded mappings between them as morphisms form a category. This category is denoted $L$-$BORN$. 

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L-valued bornologies generated by fuzzy metrics
Level bornologies of an $L$-valued bornology

Let $\mathcal{B} : 2^X \to L$ be an $L$-bornology on a set $X$, $\lambda \in L$

$$\mathcal{B}_\lambda = \{ A \in 2^X \mid \mathcal{B}(A) \geq \lambda \}$$

called $\lambda$-level bornology of the $L$-valued bornology $\mathcal{B}$.

Remark

In case when $\lambda$ is an idempotent element in the $cl$-monoid $(L, \leq, \lor, \land, \ast)$ family $\mathcal{B}_\lambda$ is a bornology on a set $X$. 
Example

\( \mathcal{B} : 2^X \rightarrow [0, 1] \), such that

\[ \forall A \subseteq X \quad \mathcal{B}(A) = \begin{cases} 
1, & \text{if } |A| < \aleph_0, \\
\frac{1}{2}, & \text{otherwise.} 
\end{cases} \]

\[ \mathcal{B}_{\frac{2}{3}} = \{A | A \subseteq X \text{ and } |A| < \aleph_0\} \]

\[ \mathcal{B}_{\frac{1}{5}} = 2^X \]
Construction of an $L$-valued bornology

Let $\mathcal{C} = \{ C_\alpha \mid \alpha \in L \}$ be a family of bornologies on a set $X$ s.t.

$$\alpha \leq \beta \implies C_\alpha \subseteq C_\beta.$$ 

For a given $A \in 2^X$ let

$$\mathcal{B}(A) = \bigvee \{ \alpha^c \mid A \in C_\alpha \}.$$ 

**Theorem**

The mapping

$$\mathcal{B} : 2^X \to L$$

constructed above is an $L$-valued bornology on $X$. Besides if $\mathcal{C}$ is uppersemicontinuous from above, then

$$\mathcal{B}_\lambda = C_{\lambda c}.$$
**Definition of fuzzy metric space**

A fuzzy metric on a set $X$ is a pair $(M, *)$ such that $M$ is a fuzzy set on $X \times X \times [0, \infty)$ and $*$ is a continuous $t$-norm satisfying the following conditions:

1. **(1FM)** $M(x, y, t) > 0$ for all $x, y \in X$, for all $t > 0$;
2. **(2FM)** $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
3. **(3FM)** $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$, for all $t > 0$;
4. **(4FM)** $M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s)$ $\forall x, y, z \in X$ $\forall t, s > 0$;
5. **(5FM)** $M(x, y, \cdot)$ is continuous for each $x, y \in X$.

The triple $(X, M, *)$ is called a fuzzy metric spaces.

A.George, P.Veeramani, On some results of fuzzy metric space, Fuzzy Sets and Systems 64(1994) 395-399.
Strong Fuzzy Metric

A fuzzy metric $M : X \times X \times [0; \infty) \rightarrow (0; 1]$ is called strong if it satisfies, in addition to the properties $(1FM)$ - $(5FM)$, the following stronger version of the axiom $(4FM)$

$$M(x, z, t) \geq M(x, y, t) \ast M(y, z, t) \quad \forall x, y, z \in X, \quad \forall t > 0$$
Example 1.

Let \( f : X \to R^+ \) be a one-to-one function and \( g : R^+ \to [0; \infty) \) be an increasing continuous function. Fixed \( \alpha, \beta > 0 \), define \( M \) by

\[
M(x, y, t) = \left( \frac{(\min\{f(x), f(y)\})^\alpha + g(t)}{(\max\{f(x), f(y)\})^\alpha + g(t)} \right)^\beta.
\]

Then, \((M, \cdot)\) is fuzzy metric on \( X \).

V.Gregori, S.Morillas, A. Sepena, Examples of fuzzy metrics and applications, Fuzzy Sets and Systems 170 (2011) 95-111
Examples of fuzzy metrics

Example 1.1.

Let $X = R^+$, and let $g$ be the identity function. Then (1) becomes

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

Then, $(M, \cdot)$ is strong fuzzy metric on $X$.

Example 2.

Let \((X, d)\) be a bounded metric space and suppose \(d(x, y) < k\) for all \(x, y \in X\). Let \(g : R^+ \rightarrow (k; +\infty)\) be an increasing continuous function. Define the function \(M\) by

\[
M(x, y, t) = 1 - \frac{d(x, y)}{g(t)}.
\]

Then \((M, \mathcal{L})\) is a fuzzy metric on \(X\).

V. Gregori, S. Morillas, A. Sepena, Examples of fuzzy metrics and applications, Fuzzy Sets and Systems 170 (2011) 95-111
Examples of fuzzy metrics

Example 2.2.

If we take $g$ as a constant function $g(t) = K > k$, then (2) becomes

$$M(x, y) = 1 - \frac{d(x, y)}{K}$$

and so $(M, L)$ is a strong fuzzy metric.

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V.Gregori, S.Morillas, A. Sepena, Examples of fuzzy metrics and applications, Fuzzy Sets and Systems 170 (2011) 95-111
Example 3.

Let $\varphi : R^+ \rightarrow [0; 1)$ be an increasing continuous function. Define the function $M$ by

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y \\ \varphi(t), & \text{otherwise.} \end{cases}$$

and so $(M, \wedge)$ is a fuzzy metric on $X$.

V.Gregori, S.Morillas, A. Sepena, Examples of fuzzy metrics and applications, Fuzzy Sets and Systems 170 (2011) 95-111
Given a fuzzy metric space \((X, M, *)\) a (crisp) topology can be introduce on \(X\) as follows: Let \((X, M, *)\) be a fuzzy metric space.

Following


given a point \(x \in X\), a positive real number \(\varepsilon\) and a non-negative real number \(t\) we define a ball at the level \(t\) with center \(x\) and radius \(\varepsilon\) as the set

\[B_{\varepsilon}(x, t) = \{y \in X \mid M(x, y, t) > 1 - \varepsilon\}.\]
Introduction
Bornological structures in the context of fuzzy sets
Category $L$ — $BORN$
Fuzzy Metric

Construction of $L$-valued bornologies from fuzzy metrics

$t \leq s \implies B_{\varepsilon}(x, t) \subseteq B_{\varepsilon}(x, s)$

$\varepsilon \leq \delta \implies B_{\varepsilon}(x, t) \subseteq B_{\delta}(x, t)$
Let \( M(x, y, t) = \frac{x \land y + t}{x \lor y + t} \quad x, y \in R^+ \quad \text{and} \quad t > 0 \)

\[
B_\varepsilon(x, t) = \left\{ y \in X \mid \frac{x \land y + t}{x \lor y + t} > 1 - \varepsilon \right\}.
\]
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\[ B_{\varepsilon} (x, \varepsilon) = \left\{ y \in X \mid \frac{x \land y + 1}{x \lor y + 1} > \varepsilon \right\} \]

\[ B_{\varepsilon} (x, \varepsilon) = \left\{ y \in X \mid \frac{x \land y + 1}{x \lor y + 1} > \frac{1}{2} \right\} \]

\[ B_{\varepsilon} (x, \varepsilon) = \left\{ y \in X \mid \frac{x \land y + 1}{x \lor y + 1} > \frac{3}{4} \right\} \]
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L-valued bornologies generated by fuzzy metrics
As different from the situation with topological structure in fuzzy metric space, the corresponding bornological structure of this space is essentially $L$-valued bornology on the powerset $2^X$. To construct an $L$-valued (where $L = [0, 1]$) bornology on the set $X$ induced by fuzzy metric $M$ we first fix a strongly decreasing bijection

$$\varphi : [0, \infty) \to (0, 1]$$

(as a typical example here one can take the hyperbola $\varphi(t) = \frac{1}{1+t} \forall t \in [0, \infty)$.)

Further we see natural approach how an $L$-valued bornology on the space $(X, M, \star)$ could be defined:
L-valued bornologies induced by fuzzy metrics

**Definition**

Given a fuzzy metric space \((X, M, \ast)\) and a number \(\alpha \in (0, 1]\) we call a set \(A \subseteq X\) \(\alpha\)-bounded if there exists \(\varepsilon > 0\) and a point \(x \in X\) such that \(A \subseteq B_\varepsilon(x, \varphi^{-1}(\alpha))\).
Given a fuzzy metric space $(X, M, *)$ let $C_\alpha$ stand for the family of all finite unions of $\alpha$-bounded subsets of $X$. One can easily verify that $C_\alpha$ is a crisp bornology on the set $X$. Besides the family $\{C_\alpha \mid \alpha \in [0, 1)\}$ is nondecreasing:

$$\alpha \leq \beta \implies C_\alpha \subseteq C_\beta.$$
Hence we can apply the construction which we have developed earlier in order to define an $L$-valued bornology $B : 2^X \to I$ from the family $\{C_\alpha \mid \alpha \in (0; 1]\}$ of crips bornologies:

$$B(A) = \bigvee \{\alpha^c \mid A \in C_\alpha\},$$

where the involution $^c : [0; 1] \to [0; 1]$ is defined in a natural way: $\alpha^c = 1 - \alpha$. 
Thank you for attention!