Necessary and possible inclusion and subsethood measure for type-2 fuzzy sets

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Outline

1. Terminology and basic definitions
2. Inclusion and subsethood measure for (ordinary) fuzzy sets
3. Inclusion and subsethood measure for interval-valued fuzzy sets
4. Inclusion and subsethood measure for type-2 fuzzy sets
Outline

1. Terminology and basic definitions
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Ordinary fuzzy sets

Let $X$ be a crisp set. A mapping

$$A : X \to [0, 1]$$

is called **fuzzy set** in a set $X$.

**Generalization of the fuzzy sets:**

- intuitionistic fuzzy sets (IFSs),
- interval-valued fuzzy sets (IVFSs),
- type-2 fuzzy sets (T2 FSs),
- type-$n$ fuzzy sets (T2 FSs).
Interval-valued fuzzy sets (IVFSs)

Let $I = [0, 1]$ and $[I]$ be the set of all closed subintervals of $I$.

A mapping

$$A : X \rightarrow [I]$$

is called an interval-valued fuzzy set in $X$ (IVFS).

For each $x \in X$, the value

$$A(x) = [A^-(x), A^+(x)] \subseteq [0, 1]$$

represents the degree of membership of an element $x$ to $A$. 
IVFSs

Terminology and basic definitions

Interval-valued fuzzy sets

\[ A(x') \]

\[ A^+(x) \]

\[ A^-(x) \]

X (universe of discourse)

membership grade

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**Type-2 fuzzy sets (T2 FSs)**

A mapping

\[ \tilde{A} : X \rightarrow [0, 1][0,1] \]

is called **type-2 fuzzy set** in a set \( X \).

For each \( x \in X \), the value

\[ \tilde{A}(x) = f_x(u) \quad f_x : [0, 1] \rightarrow [0, 1] \]

is called a **membership grade** of \( x \).

**Type-\( n \) fuzzy set** is a fuzzy set whose membership grades are fuzzy sets of type \((n - 1)\).
Terminology and basic definitions

Type-2 fuzzy sets

\( \widetilde{A} : X \rightarrow [0, 1]^{[0,1]} \)

\( \widetilde{A} : X \times [0, 1] \rightarrow [0, 1] \)

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The two dimensional plane containing all primary memberships whose secondary grades are greater than or equal to the special value $\alpha$, denoted by $\tilde{A}_\alpha$, is called $\alpha$-plane of T2 FS $\tilde{A}$, i.e.,

$$\tilde{A}_\alpha = \{(x, u) \mid \tilde{A}(x, u) \geq \alpha, x \in X, u \in [0, 1]\}.$$ 

Using vertical slice representation of T2 FS, it is easy to see that

$$\tilde{A}_\alpha = \{(\tilde{A}(x))_\alpha \mid x \in X\}$$

where $(\tilde{A}(x))_\alpha$ is $\alpha$-cut of the vertical slice $\tilde{A}(x)$.
Terminology and basic definitions

\( \alpha \)-planes

\( u \) (primary grade)

\( X \) (universe of discourse)

\( A(x,u) \) (secondary grade)

\( \alpha \)-level T2 FS or associated T2 FS

\( \alpha \)-plane

\( A(x,u) \) (secondary grade)
An $\alpha$-plane representation for T2 FS

Like ordinary FSs can be represented by $\alpha$-cuts, T2 FSs can be represented by $\alpha$-planes, i.e.,

$$\tilde{A}(x, u) = \sup\{\alpha \mid (x, u) \in \tilde{A}_\alpha\}.$$
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Inclusion and subsethood measure for (ordinary) fuzzy sets

Ordinary fuzzy sets

Inclusion:

\[ A \subseteq_F B \quad \text{iff} \quad A(x) \leq B(x), \; \forall x \in X. \]

Inclusion indicator or subsethood measure is an indicator of degree to which \( A \) is subset of \( B \), i.e., it is a mapping

\[ S : F(X) \times F(X) \rightarrow [0, 1] \]

satisfying special properties.

\[ A \nsubseteq B \quad S(A, B) = 0.9 \]
Inclusion and subsethood measure for (ordinary) fuzzy sets

Three most accepted axiomatizations:

- by Kitainik in 1987,
- by Sinha and Dougherty in 1993,
- by Young in 1996.

Young’s axioms

A mapping $S : FS(X) \times FS(X) \rightarrow [0, 1]$ is called a fuzzy subsethood measure, if $S$ satisfies the following properties (for all $A, B, C, D \in FS(X)$):

(Y1) $S(A, B) = 1$ if and only if $A \subseteq_F B$.

(Y2) Let $A^c \subseteq A$. Then $S(A, A^c) = 0$ if and only if $A = \mathcal{I}$.\(^a\)

(Y3) If $A \subseteq_F B \subseteq_F C$, then $S(C, A) \leq S(B, A)$; and if $A \subseteq_F B$, then $S(D, A) \leq S(D, B)$.

\(^a\)\(\mathcal{I}(x) = 1\), for all $x \in X$. 

\[^a\text{Note: } \mathcal{I}(x) = 1\text{, for all } x \in X.\]
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Interval-valued fuzzy sets

Inclusion:

\[ \hat{A} \subseteq_{IV} \hat{B} \quad \text{iff} \quad \hat{A}^- \subseteq_F \hat{B}^- \quad \text{and} \quad \hat{A}^+ \subseteq_F \hat{B}^+ \]

or

\[ \hat{A} \subseteq_{IV} \hat{B} \quad \text{iff} \quad \hat{A}^-(x) \leq \hat{B}^-(x) \quad \text{and} \quad \hat{A}^+(x) \leq \hat{B}^+(x) \quad \forall x \in X \]
Necessary inclusion:

\[ \hat{A} \subseteq_{N_{IV}} \hat{B} \quad \text{iff} \quad \hat{A}^+ \subseteq_{F} \hat{B}^- \]

Possible inclusion:

\[ \hat{A} \subseteq_{P_{IV}} \hat{B} \quad \text{iff} \quad \hat{A}^- \subseteq_{F} \hat{B}^+ \]
Subsethood measure is a mapping

\[ S_{IV} : IVFS(X) \times IVFS(X) \rightarrow [0, 1] \]

satisfying special properties.

Necessary subsethood measure:

\[ S_{N_{IV}}(\hat{A}, \hat{B}) = S(\hat{A}^+, \hat{B}^-) \]

Possible subsethood measure:

\[ S_{P_{IV}}(\hat{A}, \hat{B}) = S(\hat{A}^-, \hat{B}^+) \]

where \( S \) is subsethood measure for ordinary fuzzy sets.
Subsethood measure for IVFSs

**Subsethood measure** is a mapping

\[ S_{\text{IV}} : \text{IVFS}(X) \times \text{IVFS}(X) \rightarrow [0, 1] \]

satisfying special properties.

**Necessary subsethood measure:**

\[ S_{N_{\text{IV}}} (\hat{A}, \hat{B}) = S(\hat{A}^+, \hat{B}^-) \]

**Possible subsethood measure:**

\[ S_{P_{\text{IV}}} (\hat{A}, \hat{B}) = S(\hat{A}^-, \hat{B}^+) \]

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Type-2 fuzzy sets

**Inclusion** (Mizumoto and Tanaka):

\[ \tilde{A} \subseteq \tilde{B} \quad \text{iff} \quad \tilde{A}(x) \leq \tilde{B}(x), \quad \forall x \in X \]

**Alternative definition** (Hamrawi and Coupland):

\[ \tilde{A} \subseteq_{T2} \tilde{B} \quad \text{iff} \quad \tilde{A}_\alpha \subseteq_{IV} \tilde{B}_\alpha, \quad \forall \alpha \in [0, 1] \]
Inclusion and subsethood measure for type-2 fuzzy sets

**Necessary inclusion:**

\[ \tilde{A} \subseteq_{NT_2} \tilde{B} \quad \text{iff} \quad \tilde{A}_\alpha \subseteq_{IV} \tilde{B}_\alpha, \ \forall \alpha \in [0, 1] \]

**Possible inclusion:**

\[ \tilde{A} \subseteq_{PT_2} \tilde{B} \quad \text{iff} \quad \tilde{A}_\alpha \subseteq_{IV} \tilde{B}_\alpha, \ \forall \alpha \in [0, 1] \]
**Type-2 fuzzy sets**

**Subsethood measure** is a mapping

\[ S_{T2} : T2FS(X) \times T2FS(X) \rightarrow [0, 1] \]

satisfying special properties.

**Generalization of Young’s axioms**

A mapping \( S_{T2} : T2FS(X) \times T2FS(X) \rightarrow [0, 1] \) is called a fuzzy subsethood measure for T2 FSs, if

1. \( (Y1_{T2}) \) \( S_{T2}(\tilde{A}, \tilde{B}) = 1 \) if and only if \( \tilde{A} \subseteq \tilde{B} \).
2. \( (Y2_{T2}) \) Let \( \tilde{A}^c \subseteq \tilde{A} \). Then \( S_{T2}(\tilde{A}, \tilde{A}^c) = 0 \) if and only if \( A = \tilde{I} \).
3. \( (Y3_{T2}) \) If \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \), then \( S_{T2}(\tilde{C}, \tilde{A}) \leq S_{T2}(\tilde{B}, \tilde{A}) \); and if \( \tilde{A} \subseteq \tilde{B} \), then \( S_{T2}(\tilde{D}, \tilde{A}) \leq S_{T2}(\tilde{D}, \tilde{B}) \).
Procedure

1. **Decomposition:** T2 FSs into a collection of $\alpha$-planes.
2. **Computation (with $\alpha$-planes):** subsethood measure of the IVFSs.
3. **Aggregation:** weighted average.

\[
S_{T2}(\tilde{A}, \tilde{B}) = \frac{\sum_\alpha \alpha S_{IV}(\tilde{A}_\alpha, \tilde{B}_\alpha)}{\sum_\alpha \alpha}
\]
Theorem

Let $S_{IV} : IVFS(X) \times IVFS(X) \rightarrow [0, 1]$ be an subsethood measure for interval-valued fuzzy sets, i.e., $S_{IV}$ satisfies axioms (Y1$_{IV}$) - (Y3$_{IV}$). Let $\tilde{A}, \tilde{B} \in T2FS(X)$. Then a mapping

$S_{T2} : T2FS(X) \times T2FS(X) \rightarrow [0, 1]$ given by

$$S_{T2}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{IV}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha}$$

satisfies axioms (Y1$_{T2}$) - (Y3$_{T2}$).
Definition

Let $\tilde{A}, \tilde{B} \in T2FS(X)$ and let $S_{NIV}, S_{PIV}$ are necessary and possible subsethood measures for IVFSs, respectively. Then a mapping $S_{NT2} : T2FS(X) \times T2FS(X) \rightarrow [0, 1]$ given by

$$S_{NT2}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{NIV}(\tilde{A}_\alpha, \tilde{B}_\alpha)}{\sum_{\alpha} \alpha}$$

is called the necessary subsethood measure for T2 FSs $\tilde{A}$ and $\tilde{B}$. A mapping $S_{PT2} : T2FS(X) \times T2FS(X) \rightarrow [0, 1]$ given by

$$S_{PT2}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{PIV}(\tilde{A}_\alpha, \tilde{B}_\alpha)}{\sum_{\alpha} \alpha}$$

is called the possible subsethood measure for T2 FSs $\tilde{A}$ and $\tilde{B}$. 
Definition

Let $\tilde{A}, \tilde{B} \in T2FS(X)$ and let $S_{N_{IV}}, S_{P_{IV}}$ are necessary and possible subsethood measures for IVFSs, respectively. Then a mapping $S_{N_{T2}} : T2FS(X) \times T2FS(X) \rightarrow [0, 1]$ given by

$$S_{N_{T2}}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{N_{IV}}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha} = \frac{\sum_{\alpha} \alpha S(\tilde{A}_{\alpha}^{+}, \tilde{B}_{\alpha}^{-})}{\sum_{\alpha} \alpha}$$

is called the **necessary subsethood measure** for T2 FSs $\tilde{A}$ and $\tilde{B}$. A mapping $S_{P_{T2}} : T2FS(X) \times T2FS(X) \rightarrow [0, 1]$ given by

$$S_{P_{T2}}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{P_{IV}}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha} = \frac{\sum_{\alpha} \alpha S(\tilde{A}_{\alpha}^{-}, \tilde{B}_{\alpha}^{+})}{\sum_{\alpha} \alpha}$$

is called the **possible subsethood measure** for T2 FSs $\tilde{A}$ and $\tilde{B}$. 

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Thank you for your attention!