Necessary and possible inclusion and subsethood measure for type-2 fuzzy sets

Zdenko Takáč

Slovak University of Technology in Bratislava

FSTA 2012

January 30 - February 3, 2012

Liptovský Ján, Slovak Republic

Outline

1. Terminology and basic definitions
2. Inclusion and subsethood measure for (ordinary) fuzzy sets
3. Inclusion and subsethood measure for interval-valued fuzzy sets
4. Inclusion and subsethood measure for type-2 fuzzy sets

Outline

1. Terminology and basic definitions
2. Inclusion and subsethood measure for (ordinary) fuzzy sets
3. Inclusion and subsethood measure for interval-valued fuzzy sets
4. Inclusion and subsethood measure for type-2 fuzzy sets

Ordinary fuzzy sets

Let X be a crisp set. A mapping

$$A : X \to [0,1]$$

is called **fuzzy set** in a set X.

Generalization of the fuzzy sets:

- intuitionistic fuzzy sets (IFSs),
- interval-valued fuzzy sets (IVFSs),
- type-2 fuzzy sets (T2 FSs),
- type-*n* fuzzy sets (T2 FSs).

Interval-valued fuzzy sets (IVFSs)

Let $I = [0, 1]$ and $[I]$ be the set of all closed subintervals of I.

A mapping

$$A : X \to [I]$$

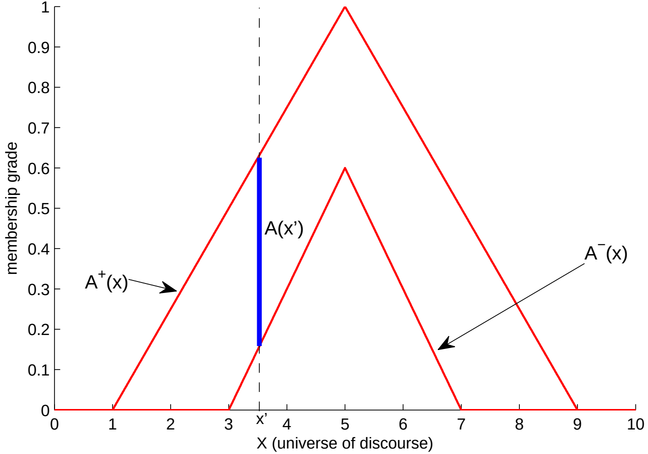
is called an **interval-valued fuzzy set** in X (**IVFS**).

For each $x \in X$, the value

$$A(x) = [A^-(x), A^+(x)] \subseteq [0, 1]$$

represents the degree of membership of an element x to A.

IVFSs


membership grade
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
x'
4
5
6
7
8
9
10
X (universe of discourse)
A(x')
A+(x)
A−(x)

Type-2 fuzzy sets (T2 FSs)

A mapping

$$\widetilde{A} : X \to [0,1]^{[0,1]}$$

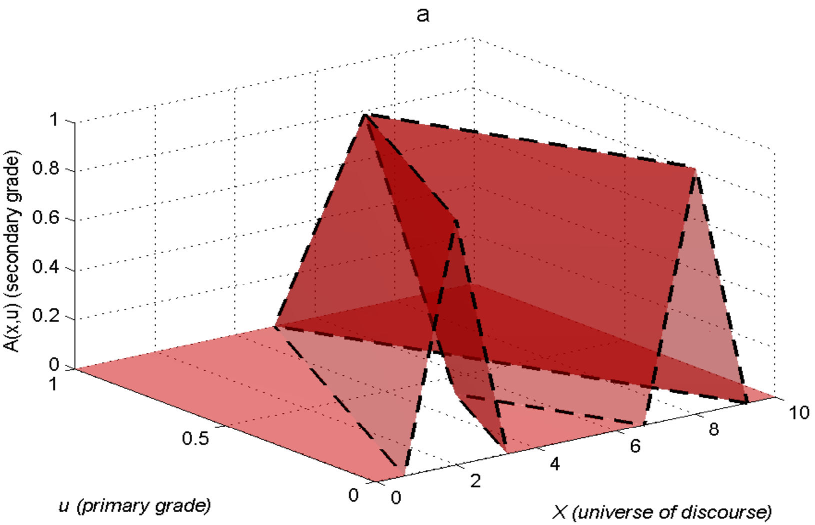
is called **type-2 fuzzy set** in a set X.

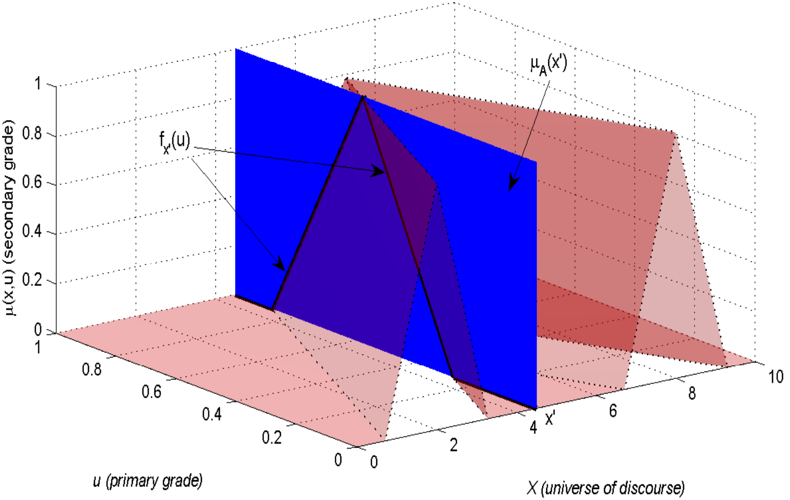
For each $x \in X$, the value

$$\widetilde{A}(x) = f_x(u) \qquad f_x : [0,1] \to [0,1]$$

is called a **membership grade** of x.

Type-n fuzzy set is a fuzzy set whose membership grades are fuzzy sets of type $(n-1)$.


a
A(x,u) (secondary grade)
1
0.8
0.6
0.4
0.2
0
1
0.5
0
u (primary grade)
0
2
4
6
8
10
X (universe of discourse)



$\widetilde{A}: X \to [0,1]^{[0,1]}$ $\qquad$ $\widetilde{A}: X \times [0,1] \to [0,1]$

α-planes

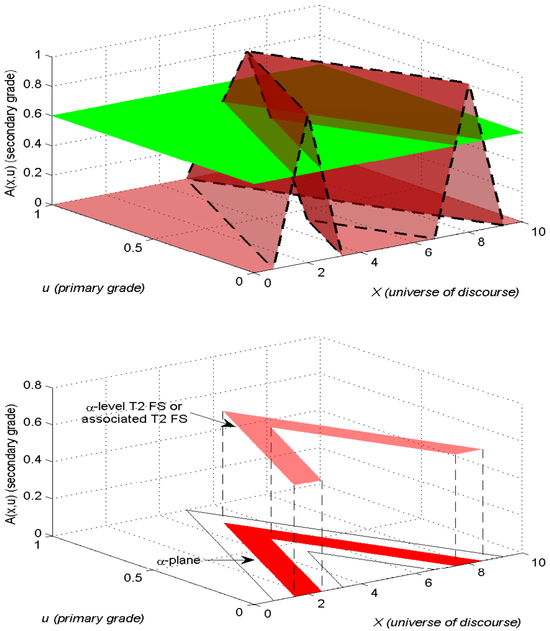
The two dimensional plane containing all primary memberships whose secondary grades are greater than or equal to the special value α, denoted by $\widetilde{A}_\alpha$, is called α*-plane* of T2 FS $\widetilde{A}$, i.e.,

$$\widetilde{A}_\alpha = \{(x,u) \,|\, \widetilde{A}(x,u) \geq \alpha, x \in X, u \in [0,1]\}.$$

Using vertical slice representation of T2 FS, it is easy to see that

$$\widetilde{A}_\alpha = \{(\widetilde{A}(x))_\alpha \,|\, x \in X\}$$

where $(\widetilde{A}(x))_\alpha$ is α-cut of the vertical slice $\widetilde{A}(x)$.


1
0.8
0.6
0.4
0.2
0
A(x,u) (secondary grade)
1
0.5
0
u (primary grade)
0
2
4
6
8
10
X (universe of discourse)
0.8
0.6
0.4
0.2
0
A(x,u) (secondary grade)
α-level T2 FS or
associated T2 FS
α-plane
1
0.5
0
u (primary grade)
0
2
4
6
8
10
X (universe of discourse)

An α-plane representation for T2 FS

Like ordinary FSs can be represented by α-cuts, T2 FSs can be represented by α-planes, i.e.,

$$\widetilde{A}(x,u) = \sup\{\alpha \,|\, (x,u) \in \widetilde{A}_\alpha\}\,.$$

Outline

Ordinary fuzzy sets

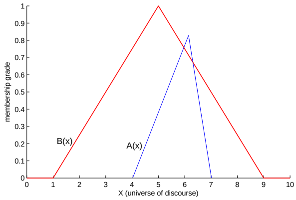
Inclusion:

$$A \subseteq_F B \qquad \text{iff} \qquad A(x) \leq B(x)\,, \ \ \forall x \in X\,.$$

Inclusion indicator or **subsethood measure** is an indicator of degree to which A is subset of B, i.e., it is a mapping

$$S : F(X) \times F(X) \to [0, 1]$$

satisfying special properties.



$A \not\subseteq B$ $\qquad S(A,B) = 0.9$

Three most accepted axiomatizations:

- by **Kitainik** in 1987,
- by **Sinha and Dougherty** in 1993,
- by **Young** in 1996.

Young's axioms

A mapping $S : FS(X) \times FS(X) \to [0,1]$ is called a fuzzy subsethood measure, if S satisfies the following properties (for all $A, B, C, D \in FS(X)$):

(Y1) $S(A,B) = 1$ if and only if $A \subseteq_F B$.

(Y2) Let $A^c \subseteq A$. Then $S(A, A^c) = 0$ if and only if $A = \mathfrak{I}$.[a]

(Y3) If $A \subseteq_F B \subseteq_F C$, then $S(C,A) \leq S(B,A)$; and if $A \subseteq_F B$, then $S(D,A) \leq S(D,B)$.

[a] $\mathfrak{I}(x) = 1$, for all $x \in X$.

Outline

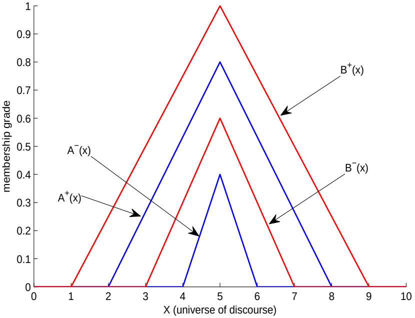
Interval-valued fuzzy sets

Inclusion:

$$\hat{A} \subseteq_{IV} \hat{B} \quad \text{iff} \quad \hat{A}^- \subseteq_F \hat{B}^- \quad \text{and} \quad \hat{A}^+ \subseteq_F \hat{B}^+$$

or

$$\hat{A} \subseteq_{IV} \hat{B} \quad \text{iff} \quad \hat{A}^-(x) \leq \hat{B}^-(x) \quad \text{and} \quad \hat{A}^+(x) \leq \hat{B}^+(x) \ \forall x \in X$$



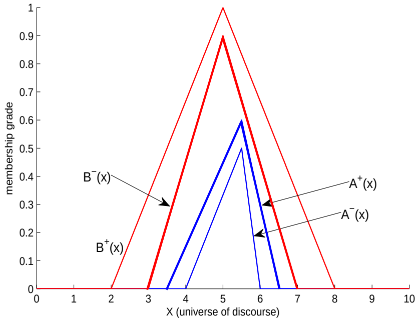
Necessary inclusion:

$$\hat{A} \subseteq_{N_{IV}} \hat{B} \quad \text{iff} \quad \hat{A}^{+} \subseteq_{F} \hat{B}^{-}$$

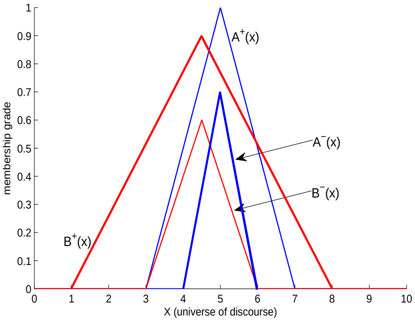
Possible inclusion:

$$\hat{A} \subseteq_{P_{IV}} \hat{B} \quad \text{iff} \quad \hat{A}^{-} \subseteq_{F} \hat{B}^{+}$$

Necessary inclusion



Possible inclusion



Subsethood measure for IVFSs

Subsethood measure is a mapping

$$S_{IV} : IVFS(X) \times IVFS(X) \to [0, 1]$$

satisfying special properties.

Necessary subsethood measure:

$$S_{N_{IV}}(\hat{A}, \hat{B}) = S(\hat{A}^+, \hat{B}^-)$$

Possible subsethood measure:

$$S_{P_{IV}}(\hat{A}, \hat{B}) = S(\hat{A}^-, \hat{B}^+)$$

where S is subsethood measure for ordinary fuzzy sets.

Subsethood measure for IVFSs

Subsethood measure is a mapping

$$S_{IV} : IVFS(X) \times IVFS(X) \to [0,1]$$

satisfying special properties.

Necessary subsethood measure:

$$S_{N_{IV}}(\hat{A}, \hat{B}) = S(\hat{A}^{+}, \hat{B}^{-})$$

Possible subsethood measure:

$$S_{P_{IV}}(\hat{A}, \hat{B}) = S(\hat{A}^{-}, \hat{B}^{+})$$

where S is subsethood measure for ordinary fuzzy sets.

Outline

Type-2 fuzzy sets

Inclusion (Mizumoto and Tanaka):

$$\tilde{A} \subseteq \tilde{B} \qquad \text{iff} \qquad \tilde{A}(x) \leq \tilde{B}(x)\,, \ \forall x \in X$$

Alternative definition (Hamrawi and Coupland):

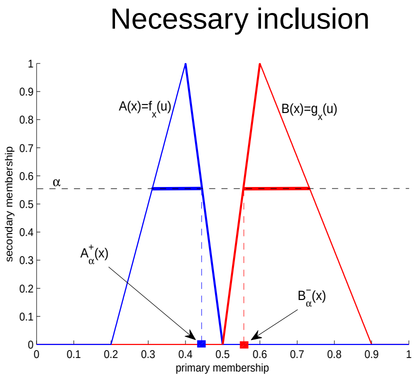
$$\tilde{A} \subseteq_{T2} \tilde{B} \qquad \text{iff} \qquad \tilde{A}_\alpha \subseteq_{IV} \tilde{B}_\alpha\,, \ \forall \alpha \in [0, 1]$$

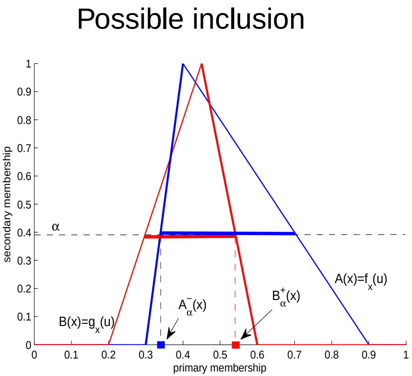
Necessary inclusion:

$$\tilde{A} \subseteq_{N_{T2}} \tilde{B} \quad \text{iff} \quad \tilde{A}_\alpha \subseteq_{N_{IV}} \tilde{B}_\alpha\,, \ \forall \alpha \in [0,1]$$

Possible inclusion:

$$\tilde{A} \subseteq_{P_{T2}} \tilde{B} \quad \text{iff} \quad \tilde{A}_\alpha \subseteq_{P_{IV}} \tilde{B}_\alpha\,, \ \forall \alpha \in [0,1]$$





Type-2 fuzzy sets

Subsethood measure is a mapping

$$S_{T2} : T2FS(X) \times T2FS(X) \to [0,1]$$

satisfying special properties.

Generalization of Young's axioms

A mapping $S_{T2} : T2FS(X) \times T2FS(X) \to [0,1]$ is called a fuzzy subsethood measure for T2 FSs, if

(Y1$_{T2}$) $S_{T2}(\tilde{A},\tilde{B}) = 1$ if and only if $\tilde{A} \subseteq \tilde{B}$.

(Y2$_{T2}$) Let $\tilde{A}^c \subseteq \tilde{A}$. Then $S_{T2}(\tilde{A},\tilde{A}^c) = 0$ if and only if $A = \tilde{\mathfrak{I}}$.

(Y3$_{T2}$) If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $S_{T2}(\tilde{C},\tilde{A}) \leq S_{T2}(\tilde{B},\tilde{A})$; and if $\tilde{A} \subseteq \tilde{B}$, then $S_{T2}(\tilde{D},\tilde{A}) \leq S_{T2}(\tilde{D},\tilde{B})$.

Procedure

1. Decomposition: T2 FSs into a collection of α-planes.
2. Computation (with α-planes): subsethood measure of the IVFSs.
3. Aggregation: weighted average.

$$S_{T2}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{IV}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha}$$

Theorem

Let $S_{IV} : IVFS(X) \times IVFS(X) \to [0,1]$ *be an subsethood measure for interval-valued fuzzy sets, i.e.,* S_{IV} *satisfies axioms (*$Y1_{IV}$*) - (*$Y3_{IV}$*). Let* $\tilde{A}, \tilde{B} \in T2FS(X)$*. Then a mapping* $S_{T2} : T2FS(X) \times T2FS(X) \to [0,1]$ *given by*

$$S_{T2}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{IV}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha}$$

satisfies axioms ($Y1_{T2}$*) - (*$Y3_{T2}$*).*

Definition

Let $\tilde{A}, \tilde{B} \in T2FS(X)$ and let $S_{N_{IV}}, S_{P_{IV}}$ are necessary and possible subsethood measures for IVFSs, respectively. Then a mapping $S_{N_{T2}} : T2FS(X) \times T2FS(X) \to [0,1]$ given by

$$S_{N_{T2}}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{N_{IV}}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha}$$

is called the **necessary subsethood measure** for T2 FSs $\tilde{A}$ and $\tilde{B}$. A mapping $S_{P_{T2}} : T2FS(X) \times T2FS(X) \to [0,1]$ given by

$$S_{P_{T2}}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha} \alpha S_{P_{IV}}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})}{\sum_{\alpha} \alpha}$$

is called the **possible subsethood measure** for T2 FSs $\tilde{A}$ and $\tilde{B}$.

Definition

Let $\tilde{A}, \tilde{B} \in T2FS(X)$ and let $S_{N_{IV}}, S_{P_{IV}}$ are necessary and possible subsethood measures for IVFSs, respectively. Then a mapping $S_{N_{T2}} : T2FS(X) \times T2FS(X) \to [0,1]$ given by

$$S_{N_{T2}}(\tilde{A}, \tilde{B}) = \frac{\sum_\alpha \alpha S_{N_{IV}}(\tilde{A}_\alpha, \tilde{B}_\alpha)}{\sum_\alpha \alpha} = \frac{\sum_\alpha \alpha S(\tilde{A}^+_\alpha, \tilde{B}^-_\alpha)}{\sum_\alpha \alpha}$$

is called the **necessary subsethood measure** for T2 FSs $\tilde{A}$ and $\tilde{B}$. A mapping $S_{P_{T2}} : T2FS(X) \times T2FS(X) \to [0,1]$ given by

$$S_{P_{T2}}(\tilde{A}, \tilde{B}) = \frac{\sum_\alpha \alpha S_{P_{IV}}(\tilde{A}_\alpha, \tilde{B}_\alpha)}{\sum_\alpha \alpha} = \frac{\sum_\alpha \alpha S(\tilde{A}^-_\alpha, \tilde{B}^+_\alpha)}{\sum_\alpha \alpha}$$

is called the **possible subsethood measure** for T2 FSs $\tilde{A}$ and $\tilde{B}$.

Thank you for your attention!