Aggregation Functions and the Associativity in the Sense of Post

Andrea Stupňanová [1], Anna Kolesárová [2]

[1]Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, Slovak University of Technology in Bratislava

[2]Institute IAM, Faculty of Chemical and Food Technology
Slovak University of Technology in Bratislava

FSTA 2012, Liptovský Ján, Slovakia,
30 January – 3 February 2012

An aggregation function $A : [0,1]^n \to [0,1]$, $n \geq 2$

monotonicity $A(\mathbf{x}) \leq A(\mathbf{y})$ whenever $\mathbf{x}, \mathbf{y} \in [0,1]^n, \mathbf{x} \leq \mathbf{y}$

boundary conditions $A(\mathbf{0}) = 0$, $A(\mathbf{1}) = 1$

? Under which constrains is an associative n-ary aggregation function $A : [0,1]^n \to [0,1]$ an extension of a binary associative aggregation function $B : [0,1]^2 \to [0,1]$?

Contents

A binary function $G : [0,1]^2 \to [0,1]$

associative

$$G(x, G(y,z)) = G(G(x,y), z) \text{ for all } x, y, z \in [0,1] \quad (1)$$

has a neutral element $e \in [0,1]$

$$G(x,e) = G(e,x) = x \text{ for all } x \in [0,1] \quad (2)$$

symmetric

$$G(x,y) = G(y,x) \text{ for all } x, y \in [0,1] \quad (3)$$

An aggregation function $G : [0,1]^2 \to [0,1]$ is called

- *a triangular norm (t-norm)* if it is associative, symmetric and it has neutral element $e = 1$
- *a triangular conorm (t-conorm)* if it is associative, symmetric and it has neutral element $e = 0$
- *a uninorm* if it is associative, symmetric and it has neutral element $e \in]0, 1[$
- *a copula* if it has neutral element $e = 1$ and it is 2-increasing, i.e.,

$$G(x', y') - G(x, y') - G(x', y) + G(x, y) \geq 0 \tag{4}$$

for all $x, y, x', y' \in [0, 1]$, $x \leq x'$, $y \leq y'$.

Definition (Post)

Let $n \geq 2$. A function $F\colon [0,1]^n \to [0,1]$ is said to be associative whenever for all $x_1, \dots, x_n, \dots, x_{2n-1} \in [0,1]$ it holds

$$\begin{aligned} & F\left(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}\right) = \\ = & F\left(x_1, F(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1}\right) = \\ = & \cdots = F\left(x_1, \dots, x_{n-1}, F(x_n, \dots, x_{2n-1})\right). \end{aligned} \tag{5}$$

Definition

Let $n \geq 2$. A function $F : [0,1]^n \to [0,1]$ is said to have neutral element $e \in [0,1]$ whenever $F(x_1, \dots, x_n) = x_i$ if $x_j = e$ for each $j \neq i$.

Definition

Let $n \geq 2$. A function $F : [0,1]^n \to [0,1]$ is called symmetric whenever for each $\mathbf{x} \in [0,1]^n$ and each permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ it holds

$$F(x_1,\dots,x_n) = F(x_{\sigma(1)},\dots,x_{\sigma(n)}).$$

We say that a function F is an n-ary extension of a binary function G if it holds

$$F(x_1,\dots,x_n) = G(G(\dots G(G(x_1,x_2),x_3)\dots),x_{n-1}),x_n)$$

for all n-tuples in $[0,1]^n$.

Example

(i) Define a mapping $F\colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x_1, x_2, x_3) = x_1 - x_2 + x_3$. Then F is a ternary associative function. Observe that there is no binary associative function whose ternary extension coincides with F. Moreover, F has no neutral element and it is not symmetric.

(ii) Let $C\colon [0,1]^3 \to [0,1]$ be given by $C(x_1, x_2, x_3) = x_1 \min\{x_2, x_3\}$. Then $e = 1$ is neutral element of C, but C is not associative. Note that C is a ternary copula which is not symmetric.

Contents

Theorem (Stupňanová & Kolesárová, AGOP 2011)

Consider $n \geq 2$. Let $e \in [0, 1]$. Then the following claims are equivalent:

(i) *A mapping $F\colon [0, 1]^n \to [0, 1]$ is an associative function with neutral element e.*

(ii) *There is a binary associative function $G\colon [0, 1]^2 \to [0, 1]$ with neutral element e whose n-ary extension is F.*

Theorem shows that under the neutral element existence, the associativity of n-ary functions is classically related to the associativity of binary functions.

Contents

Definition

Let $n \geq 2$. An aggregation function $A : [0,1]^n \to [0,1]$ which is associative (in the sense of Post), symmetric, and possesses a neutral element $e \in [0,1]$ is called:

- *an n-ary t-norm* if $e = 1$;
- *an n-ary t-conorm* if $e = 0$;
- *an n-ary uninorm* if $e \in]0,1[$.

Corollary

Let $n > 2$. A function $A : [0,1]^n \to [0,1]$ is an n-ary t-norm (t-conorm, uninorm) if and only if there is a binary t-norm (t-conorm, uninorm) $B : [0,1]^2 \to [0,1]$ such that A is an n-ary extension of B.

n-dimensional copula (*n*-copula)

For $n \geq 2$, a function $C\colon [0,1]^n \to [0,1]$

(C1) $C(x_1, \ldots, x_n) = x_i$ whenever $\forall j \neq i,\ x_j = 1$;

(C2) $C(x_1, \ldots, x_n) = 0$ whenever $0 \in \{x_1, \ldots, x_n\}$;

(C3) the *n*-increasing property, i.e.,
$\forall\, \mathbf{x}, \mathbf{y} \in [0,1]^n,\ x_i \leq y_i,\ i = 1, \ldots, n$, it holds

$$\sum_{J \subset \{1,\ldots,n\}} (-1)^{|J|} C\left(u_1^J, \ldots, u_n^J\right) \geq 0, \text{ where } u_i^J = \begin{cases} x_i, & \text{if } i \in J, \\ y_i, & \text{if } i \notin J. \end{cases} \tag{6}$$

Each *n*-ary copula is an *n*-ary aggregation function with a neutral element $e = 1$.

There are two distinguished functions which are n-copulas for each $n \geq 2$: the so-called *minimum* n-copula M and the *product* n-copula Π, given by

$$\begin{aligned} M(x_1, \ldots, x_n) &= \min\{x_1, \ldots, x_n\}, \\ \Pi(x_1, \ldots, x_n) &= \prod_{i=1}^{n} x_i. \end{aligned}$$

The minimum n-copula M describes the comonotone dependence of random variables $X_1, \ldots, X_n$ and the product n-copula Π describes their independence.

For each n-copula C it holds

$$W \leq C \leq M,$$

where W is the so-called Fréchet-Hoeffding lower bound, given by

$$W(x_1, \ldots, x_n) = \max\left\{0, \sum_{i=1}^{n} x_i - (n-1)\right\}.$$

It is a well-known fact that this function is a copula only for $n = 2$, and in that case describes the countermonotone dependence of random variables X_1 and X_2.

Indeed, $W(x_1, x_2, x_3) = \max(0, x_1 + x_2 + x_3 - 2)$, and considering $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\mathbf{y} = (1, 1, 1)$, we see that

$$W(1,1,1) - W\left(\frac{1}{2},1,1\right) - W\left(1,\frac{1}{2},1\right) - W\left(1,1,\frac{1}{2}\right) +$$

$$+W\left(\frac{1}{2},\frac{1}{2},1\right)+W\left(\frac{1}{2},1,\frac{1}{2}\right)+W\left(1,\frac{1}{2},\frac{1}{2}\right)-W\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = -\frac{1}{2} \ngeq 0,$$

proving that ternary W is not a copula.

All the three basic 2-copulas (copulas, for short) M, Π and W are associative.

An Archimedean copula

Let $C\colon [0,1]^2 \to [0,1]$ be an associative copula satisfying $C(x,x) < x$ for all $x \in]0,1[$. Then C is called an Archimedean copula.

Theorem (Moynihan, 1978)

A function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous strictly decreasing convex function $f\colon [0,1] \to [0,\infty]$, $f(1) = 0$, such that

$$C(x_1, x_2) = f^{(-1)}\left(f(x_1) + f(x_2)\right), \tag{7}$$

where $f^{(-1)}$ is the pseudo-inverse of f.

Recall that the pseudo-inverse $f^{(-1)}\colon [0,\infty] \to [0,1]$ is given by

$$f^{(-1)}(u) = f^{-1}\left(\min(f(0), u)\right).$$

Copulas W and Π are Archimedean, with generators f_W and f_Π, respectively, given by $f_W(x) = 1 - x$ and $f_\Pi(x) = -\log x$. If we define the function $f_{(1)} \colon [0,1] \to [0,\infty]$ by $f_{(1)}(x) = \frac{1}{x} - 1$, it is also a generator and the corresponding Archimedean copula $C_{(1)} \colon [0,1]^2 \to [0,1]$ is given by

$$C_{(1)}(x_1, x_2) = \frac{x_1\, x_2}{x_1 + x_2 - x_1 x_2}$$

whenever $(x_1, x_2) \neq (0,0)$.

For a general associative copula C we have the next representation theorem

Theorem

A function $C\colon [0,1]^2 \to [0,1]$ is an associative copula if and only if there is a system $(\,]a_k, b_k[\,)_{k\in\mathcal{K}}$ of pairwise disjoint open subintervals of $[0,1]$ and a system $(C_k)_{k\in\mathcal{K}}$ of Archimedean copulas such that

$$C(x_1,x_2) = \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{x_1-a_k}{b_k-a_k}, \frac{x_2-a_k}{b_k-a_k}\right), & \text{if } (x_1,x_2) \in\,]a_k,b_k[^2 \\ & \text{for some } k\in\mathcal{K}, \\ M(x_1,x_2), & \text{else.} \end{cases} \tag{8}$$

Copula C given by is called an ordinal sum copula, with notation $(\langle a_k, b_k, C_k\rangle \mid k \in \mathcal{K})$.

Example

Let $C = \left(\langle 0, \frac{1}{2}, \Pi \rangle\right)$. Then

$$C(x_1, x_2) = \begin{cases} 2x_1 x_2, & \text{if } (x_1, x_2) \in \left]0, \frac{1}{2}\right[^2, \\ M(x_1, x_2), & \text{else.} \end{cases}$$

Based on previous theorems and recent results on ordinal sum structure of n-copulas proved by Mesiar and Sempi [2010], we have the next result.

Corollary

Let $n \geq 2$. A function $C\colon [0,1]^n \to [0,1]$ is an associative n-copula if and only if there is a system $(\,]a_k, b_k[\,)_{k\in\mathcal{K}}$ of pairwise disjoint open subintervals of $]\,0,1\,[$, and a system $(C_k)_{k\in\mathcal{K}}$ of associative n-copulas satisfying the diagonal inequality $C_k(x,\dots,x) < x$ for all $x \in]\,0,1\,[$ and $k \in \mathcal{K}$ such that

$$C(x_1,\dots,x_n) = \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{\min\{x_1,b_k\}-a_k}{b_k-a_k},\dots,\frac{\min\{x_n,b_k\}-a_k}{b_k-a_k}\right), \\ \qquad \text{if } \min\{x_1,\dots,x_n\} \in]a_k,b_k[\quad \text{for some } k \in \mathcal{K}, \\ M(x_1,\dots,x_n), \qquad \text{else.} \end{cases} \tag{9}$$

To complete the representation of associative n-copulas, the characterization of such copulas satisfying the diagonal inequality is necessary.

Theorem (Stupňanová, Kolesárová, Kybernetika 47 (2011))

Let $n \geq 2$. A function $C\colon [0,1]^n \to [0,1]$ is an associative n-copula satisfying the diagonal inequality $C(x,\ldots,x) < x$ for all $x \in]\,0,1\,[$ if and only if there is a generator f whose pseudo-inverse $f^{(-1)}$ is an $(n-2)$-times differentiable function with derivatives alternating the sign, such that $(-1)^n \frac{\mathbf{d}^{n-2} f^{(-1)}}{\mathbf{d}\, x^{n-2}}$ is a convex function, and

$$C(x_1,\ldots,x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right). \qquad (10)$$

McNeil, Nešlehová [2009]

Example

As already mentioned, the product n-copula Π is associative for any $n \geq 2$. Evidently, $\Pi(x, \ldots, x) = x^n < x$ whenever $x \in]\,0, 1\,[$. As the generator f_Π of the copula Π is given by $f_\Pi(x) = -\log x$, it holds $f_\Pi^{(-1)}(x) = f_\Pi^{-1}(x) = e^{-x}$, hence for any k,

$$\frac{\mathbf{d}^k f_\Pi^{-1}(x)}{\mathbf{d}\, x^k} = (-1)^k e^{-x}.$$

Derivatives alternate the sign and for any $n \geq 2$,

$$(-1)^n \frac{\mathbf{d}^{n-2} f_\Pi^{(-1)}(x)}{\mathbf{d}\, x^{n-2}} = e^{-x}$$

is a convex function.

Example

A similar result can be shown for the generator $f_{(1)}$ introduced in this section, given by $f_{(1)}(x) = \frac{1}{x} - 1$. It holds $f_{(1)}^{(-1)}(x) = f_{(1)}^{-1}(x) = (1+x),^{-1}$ which implies that

$$(-1)^n \frac{\mathbf{d}^{n-2} f_{(1)}^{(-1)}(x)}{\mathbf{d}\, x^{n-2}} = (n-2)!\,(1+x)^{-n+1}$$

is convex. The corresponding n-copula $C_{(1)}$ is given by

$$C_{(1)}(x) = \left(\sum_{i=1}^{n} \frac{1}{x_i} - (n-1) \right)^{-1}.$$

Example

The weakest associative n-copula is the Clayton copula $C_{(-\frac{1}{n-1})}$ generated by the generator $f_{(-\frac{1}{n-1})} \colon [0,1] \to [0,\infty]$, $f_{(-\frac{1}{n-1})} = 1 - x^{\frac{1}{n-1}}$. The corresponding pseudo-inverse $f^{(-1)}_{(-\frac{1}{n-1})} \colon [0,\infty] \to [0,1]$ is given by

$$f^{(-1)}_{(-\frac{1}{n-1})}(x) = \begin{cases} (1-x)^{n-1}, & \text{if } x \le 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Then $(-1)^n \dfrac{\mathbf{d}^{n-2} f^{(-1)}_{(-\frac{1}{n-1})}(x)}{\mathbf{d}\, x^{n-2}} = (n-1)! \max\{1-x, 0\}$ is convex but not differentiable.

Example

The function $C\colon [0,1]^n \to [0,1]$ given by

$$C(x_1, \dots, x_n) = \begin{cases} 2^{n-1} \prod_{i=1}^{n} \min\left\{x_i, \frac{1}{2}\right\}, & \text{if } \min\{x_1, \dots, x_n\} < \frac{1}{2}, \\ M(x_1, \dots, x_n), & \text{else}, \end{cases} \tag{11}$$

is an n-ary extension of the ordinal sum copula $\left(\langle 0, \frac{1}{2}, \Pi \rangle\right)$. As n-ary function Π is an associative n-copula for each $n \geq 2$, our function C given by (11) is also an associative n-copula for each $n \geq 2$.

Contents

Problem (Problem 2.1(Mesiar),Open Problems at FSTA 2010)

Is there a representation of n-ary associative copulas (in the sense of Post) similar to the concerning binary copulas?

Associative n-copulas are just n-ary extensions of appropriate associative copulas.

Note that not each n-ary aggregation function A associative in the sense of Post should possess a neutral element. For example, for each $a \in [0, 1]$, the function $A : [0, 1]^3 \to [0, 1]$ given by

$$A(x, y, z) = \mathrm{med}(x, a, y, a, z)$$

is associative (and symmetric), but it has a neutral element only if $a \in \{0, 1\}$. Hence the complete characterization of n-ary associative aggregation functions which can be seen as n-ary extensions of binary associative aggregation functions is still an open problem.

References

M. Couceiro (2010). On two generalizations of associativity. *In: Abstracts of FSTA 2010 (E. P. Klement et. al., eds.), Liptovský Ján 2010*, pp 47.

T. Calvo, A. Kolesárová, M. Komorníková and R. Mesiar (2002). Aggregation operators: properties, classes and construction methods, in *Aggregation Operators. New Trends and Applications*, eds. T. Calvo, G. Mayor and R. Mesiar (Physica-Verlag, Heidelberg) pp. 3–104.

J. Fodor, R. Yager, and A. Rybalov (1997). Structure of uninorms. *Internat. J. Uncertainty, Fuzziness Knowledge-Based Systems* **5** , pp 411–427.

M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap (2009). Aggregation Functions. Cambridge University Press, Cambridge.

R. Mesiar and P. Sarkoci (2010). Open problems posed at the 10th International Conference on Fuzzy Set Theory and Appl. (FSTA 2010), Liptovský Ján. *Kybernetika* **46**, pp 585–598.

R. Mesiar and C. Sempi (2010). Ordinal sums and idempotents of copulas. *Aequationes Math.* **79**, pp 1–2, 39–52.

R. Moynihan (1978). Infinite τ_T products of distribution functions. *J. Austral. Math. Soc. Ser. A* **26**, pp 227–240.

E.-L. Post (1940). Polyadic groups. *Trans. Amer. Math. Soc.* **48**, pp 208–350.