Various Versions of Weak Reflexivity of Fuzzy Relations and Applications

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Abstract

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We use the lattice valued approach to fuzzy sets and fuzzy relations, with the co-domain being a complete lattice (or sometimes a residuated lattice L). If $1 \in L$ is the top element of the lattice, then a usual concept of reflexivity of a fuzzy relation $R : A \times A \longrightarrow L$ is that for every $x \in A$, $R(x,x) = 1$. This is a rather strong requirement, so there are several attempts to weaken this property.
(1) $R(x,y) \leq R(x,x)$ and $R(x,y) \leq R(y,y)$ for every $x, y \in A$.
(2) $\bigvee_{x \in A} R(x,y) = 1$ for every $y \in A$ and $\bigvee_{y \in A} R(x,y) = 1$ for every $x \in A$.

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The third way of weakening the reflexivity is the concept of fuzzy reflexivity of a fuzzy relation on a fuzzy set. A fuzzy relation $R : A^2 \to L$ on A is said to be a **fuzzy relation on** μ if for all $x, y \in A$ $R(x, y) \leq \mu(x) \wedge \mu(y)$. Consequently, a fuzzy relation R on a fuzzy set μ is **reflexive** if the following holds:
for all $x \in A$, $R(x, x) = \mu(x)$.
In this work, connections between various concepts of weak reflexivity are given, taking into account various types of co-domains. Weak reflexivity is also connected with transitivity and idempotency.

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3. The use of weak reflexivity in a definition of the fuzzy lattice (equivalence of two definitions of fuzzy lattices).

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1. A role of weak reflexivity in solutions of fuzzy equations and inequalities (fuzzy sets closed under fuzzy relations).
2. The use of weak reflexivity in compatible fuzzy equalities, by which fuzzy identities are introduced. It is connected to a definition of fuzzy algebraic structures in a new framework.
3. The use of weak reflexivity in a definition of the fuzzy lattice (equivalence of two definitions of fuzzy lattices).
4. Lattice representation problems. Namely, having in mind that the lattice of all fuzzy weak congruences is an algebraic lattice, we tackle a representation problem of algebraic lattices by a lattice of weak fuzzy congruences on an algebra, which is still an open problem even in the crisp case.

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3. R is (weak) reflexive if for every $x, y \in X$,

$$R(x,x) \geq R(x,y)$$

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L. A. Zadeh, K.-S. Fu, K. Tanaka, M. Shimura, Fuzzy sets and their applications to cognitive and decision processes. Academic Press, New York, 1975

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A fuzzy relation on X is a fuzzy set on X^2.
A p-**cut** of a fuzzy relation on X is an ordinary relation on X: for $p \in L$,

$$R_p = \{(x, y) \in X^2 \mid R(x, y) \geq p\}.$$

Fuzzy relations

$(a-w)$ R is **antisymmetric** if for all $x, y \in X$,
if $x \neq y$ then $R(x, y) \wedge R(y, x) = 0$.
$(a-s)$ R is **antisymmetric** if for all $x, y \in X$,
if $x \neq y$ then either $R(x, y) = 0$ or $R(y, x) = 0$.

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(t) R is **transitive** if for all $x, z, y \in X$,
$R(x, y) \wedge R(y, z) \leq R(x, z)$.
(s) R is **symmetric** if for all $x, y \in X$, $R(x, y) = R(y, x)$.

Fuzzy set closed wrt fuzzy relations

Let $\mu : X \rightarrow L$ be a fuzzy set and $R : X^2 \rightarrow L$ a fuzzy relation. Then μ is said to be **closed with respect to** R if for every $x, y \in X$

$$\mu(x) \wedge R(x, y) \leq \mu(y).$$

Fuzzy set closed wrt fuzzy relations

In the following we denote by $\mathcal{S}_R$ the collection of all fuzzy sets closed under a fuzzy relation R on a universe X:

$$\mathcal{S}_R = \{\mu \in \mathcal{F}(X) \mid \mu(x) \wedge R(x,y) \leq \mu(y) \text{ for any } x, y \in X\}.$$

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The poset $(\mathcal{S}_R, \subseteq)$ is a complete lattice.

Fuzzy inequations and equations

Based on a fuzzy control problem, we are interested in the identification of solutions μ of the inequation

$$\bigvee_{x \in X} (\mu(x) \wedge R(x, y)) \leq \mu(y) \tag{1}$$

and of the equation

$$\bigvee_{x \in X} (\mu(x) \wedge R(x, y)) = \mu(y), \tag{2}$$

where R is a given fuzzy binary relation on a universe X. The inequation and the equation are supposed to be fulfilled for any $y \in X$, and μ represents a fuzzy subset of X.

Given a solution μ of inequation (1), that is, given an element $\mu \in \mathcal{S}_R$, we have a chain of solutions of the same inequation:

$$\mu \supseteq \mu_1 \supseteq \mu_2 \supseteq \ldots \supseteq \mu_{n-1} \supseteq \mu_n \supseteq \ldots$$

where $\mu_n \in \mathcal{S}_R$ for every n, and

$$\mu_n(x) = \bigvee_{z \in X} (\mu_{n-1}(z) \wedge R(z, x))$$

for every $x \in X$. If two members of this chain are equal, i.e., if for some n, $\mu_{n-1} = \mu_n$, then μ_n is a solution of the equation:

$$\bigvee_{x \in X} (\mu_n(x) \wedge R(x, y)) = \mu_n(y).$$

Fuzzy inequations and equations

Proposition

Let R be a fuzzy binary relation on X and ν the fuzzy subset on X defined by

$$\nu(x) = R(x, x).$$

Then, ν is a solution of the inequation (1) *if and only if it is a solution of the equation* (2).

Proposition

Let μ be an element in $\mathcal{S}_R$ and let $\mathcal{S}_R^\mu$ be the subset of $\mathcal{S}_R$ defined by

$$\mathcal{S}_R^\mu := \{\mu_n \in \mathcal{F}(X) \mid n \in \mathbb{N} \text{ and for all } x \in X, \\ \mu_n(x) = \bigvee_{z \in X} (\mu_{n-1}(z) \wedge R(z,x)), \text{ with } \mu_0 = \mu\}.$$

Then the fuzzy subset $\bar{\mu} \in \mathcal{F}(X)$ defined by

$$\bar{\mu}(x) = \mu(x) \wedge R(x,x) \text{ for every } x \in X$$

is a lower bound of $\mathcal{S}_R^\mu$ in the poset $(\mathcal{F}(X), \subseteq)$.

Proposition

Let R be a fuzzy binary relation on X and ν the fuzzy subset on X defined by

$$\nu(x) = R(x, x).$$

Then, for any $\mu \in \mathcal{S}_R$, $\bar{\mu}$ belongs to $\mathcal{S}_R$ if and only if $\nu \in \mathcal{S}_R$.

Corollary

Let R be a fuzzy binary relation on X. For any $\mu \in \mathcal{S}_R$, we have that

$$\bar{\mu} \in \mathcal{S}_R \text{ if and only if for all } x, y \in X, R(x,x) \wedge R(x,y) \leq R(y,y).$$

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Corollary

Let R be a fuzzy binary relation on X. For any $\mu \in \mathcal{S}_R$, we have that

$\bar{\mu} \in \mathcal{S}_R$ *if and only if for all* $x, y \in X, R(x,x) \wedge R(x,y) \leq R(y,y)$.

Corollary

Let R be a fuzzy binary relation on X fulfilling the following weak-reflexivity condition:

$$\text{For all } x, y \in X,\ R(x,y) \leq R(y,y).$$

Then, for any $\mu \in \mathcal{S}_R$, the lower bound $\bar{\mu}$ of $\mathcal{S}_R^{\mu}$ belongs to $\mathcal{S}_R$.

Fuzzy inequations and equations

Theorem

If R is a fuzzy binary relation on a set X, then (the characteristic function of)

$$X \text{ is a solution of equation } \bigvee_{x \in X} (\mu(x) \wedge R(x,y)) = \mu(y)$$

$$\text{if and only if } \bigvee_{x \in X} R(x,y) = 1 \text{ for every } y \in X .$$

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For a nullary operation (constant) $c \in F$, we require that

$$\mu(c) = 1, \tag{4}$$

where 1 is the greatest (the top) element in L.

A fuzzy relation $R : A^2 \rightarrow L$ on the underlying set A of an algebra $\mathcal{A} = (A, F)$ is said to be **compatible** with the operations, if for every n-ary operation $f \in F$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$$R(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \geqslant \bigwedge_{i=1}^{n} R(x_i, y_i). \tag{5}$$

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In particular, for $n = 1$, $R(f(x_1), f(y_1)) \geqslant R(x_1, y_1)$.
A reflexive, symmetric and transitive relation on A, which is compatible with fundamental operations on $\mathcal{A}$, is a **fuzzy congruence** on this algebra. Obviously, **compatible fuzzy equalities** which additionally satisfy property (3) are also special fuzzy congruences.

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This condition is a generalization of the following crisp relational property: R is a binary relation on a subset M of A, if xRy implies $x, y \in M$.

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This condition is a generalization of the following crisp relational property: R is a binary relation on a subset M of A, if xRy implies $x, y \in M$.
Due to this condition, it is not possible to define reflexivity in the usual way. Therefore, we introduce the following definition.

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A fuzzy equivalence relation R on μ, fulfilling for all $x, y \in A$ $x \neq y$,:

$$\text{if } R(x,x) \neq 0, \text{ then } R(x,x) > R(x,y) \text{ and } R(x,x) > R(y,x), \tag{8}$$

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A fuzzy relation R on μ is *compatible* with the operations on this fuzzy subalgebra if it fulfils the property (5).

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is called a **fuzzy equality** relation on a fuzzy set μ.
A fuzzy relation R on μ is *compatible* with the operations on this fuzzy subalgebra if it fulfils the property (5). A compatible fuzzy equivalence on μ is a **fuzzy congruence** on this fuzzy subalgebra.

We use this framework to develop notions of a **fuzzy identity**, **fuzzy equational class**, **fuzzy variety**, and other notions from fuzzy universal algebra.

Fuzzy lattice

Fuzzy weak ordering relations

A fuzzy relation $R : X^2 \to L$ is a **fuzzy weak ordering relation** (lattice valued weak ordering relation) on X if the following holds:

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antisymmetry: for all $x, y \in X$, if $x \neq y$, then $R(x,y) \wedge R(y,x) = 0$;

transitivity: for all $x, y, z \in X$, $R(x,y) \geqslant R(x,z) \wedge R(z,y)$.

Fuzzy lattice as a fuzzy algebra

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if for all $x, y \in L$,

$$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y),$$

$$\mu(x \vee y) \geq \mu(x) \wedge \mu(y).$$

Fuzzy lattice as a fuzzy relation

Let M be a nonempty set and $L' = 0 \oplus L$ be a complete lattice with the least element 0 and a unique atom 0_L. Then the mapping

$$\overline{\rho} : M^2 \longrightarrow L'$$

is an L–fuzzy lattice (as a fuzzy relation) if the following hold:

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1. $\overline{\rho}$ is a weak L-fuzzy ordering relation.

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is an L–fuzzy lattice (as a fuzzy relation) if the following hold:

1. $\overline{\rho}$ is a weak L-fuzzy ordering relation.
2. For all $x, y \in M$ there exists $S \in M$, such that for all $p \in \{0_L\} \cup \{p \in L \mid x, y \in N_p\}$, the following holds:

$$\overline{\rho}(x, S) \geq p, \quad \overline{\rho}(y, S) \geq p \tag{9}$$

and for all $s \in M$ the following implication holds:

$$\overline{\rho}(x, s) \geq p \wedge \overline{\rho}(y, s) \geq p \Rightarrow \overline{\rho}(S, s) \geq p. \tag{10}$$

Fuzzy lattice as a fuzzy relation (cont.)

3. For all $x, y \in M$ there exists $l \in M$, such that for all $p \in \{0_L\} \cup \{p \in L \mid x, y \in N_p\}$, the following holds:

$$\overline{\rho}(l, x) \geq p, \quad \overline{\rho}(l, y) \geq p \tag{11}$$

and for all $i \in M$ the following implication holds:

$$\overline{\rho}(i, x) \geq p \wedge \overline{\rho}(i, y) \geq p \Rightarrow \overline{\rho}(i, l) \geq p. \tag{12}$$

Theorem

A relation $R : S^2 \longrightarrow L$ is an L-fuzzy weak ordering relation if and only if all cuts except 0-cut are weak ordering relations.

Demonstration of cutworthy properties

Theorem

A relation $R : S^2 \longrightarrow L$ is an L-fuzzy weak ordering relation if and only if all cuts except 0-cut are weak ordering relations.

Theorem

Fuzzy set $R : L \longrightarrow M$ is a fuzzy lattice if and only if all cuts are sublattices of L.

Theorem

Let $\overline{M} : M \to L$ be an L–fuzzy lattice, where $(M, \vee, \wedge)$ is a lattice and let $L' = 0 \oplus L$. Then, the mapping $\overline{\rho} : M^2 \to L'$ defined by

$$\overline{\rho}(x,y) = \begin{cases} \overline{M}(x) \wedge_L \overline{M}(y), & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

is an L–fuzzy lattice (as an L–fuzzy relation). Moreover, M_p and (N_p, ρ_p), for $p \in L$ are the same crisp sublattices of M.

Theorem

Let $\overline{\rho} : M^2 \longrightarrow L'$ be an $L-$ fuzzy lattice (as a fuzzy relation), where $L' = 0 \oplus L$ is a complete lattice with a unique atom 0_L, the top element 1_L and the bottom element 0. Then, the mapping $\overline{M} : M \longrightarrow L$ defined by:

$$\overline{M}(x) = \overline{\rho}(x, x)$$

is an L–fuzzy lattice (as an L–fuzzy algebra, taking (M, ρ_{0_L}) as the corresponding crisp lattice. Moreover, M_p and (N_p, ρ_p) are the same sublattices of M.

Weak congruence relations

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Weak reflexivity: for every nullary operation c on algebra, $(c, c) \in R$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con}_w(\mathcal{A})$.

The congruence lattice $\mathrm{Con}(\mathcal{A})$ *of* $\mathcal{A}$ *is a principal filter in* $\mathrm{Con}_w(\mathcal{A})$*, generated by the diagonal relation* Δ *of* A*.*

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The subalgebra lattice $\mathrm{Sub}(\mathcal{A})$ *is isomorphic to the principal ideal generated by* Δ, *by sending each weak congruence* θ *contained in* Δ *to its domain*

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Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice.

Theorem

(Czédli, Erné, Šešelja, Tepavčević, 2009) *A group is a Dedekind group if and only if its weak congruence lattice is modular.*

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Corollary

(Czédli, Erné, Šešelja, Tepavčević, 2009) *A group is locally cyclic if and only if its weak congruence lattice is distributive.*

Representation of lattices by weak congruences

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Bacic representation problem

Represent an algebraic lattice by the weak congruence lattice of an algebra.

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Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\mathrm{Con}\,\mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
Obviously, $\mathrm{Con}_{\mathrm{w}}(\mathcal{A}) \cong \mathrm{Con}\,\mathcal{B} \cong L$.

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Obviously, $\mathrm{Con_w}(\mathcal{A}) \cong \mathrm{Con}\,\mathcal{B} \cong L$.

The above construction by which the diagonal relation of the algebra corresponds to the bottom of the lattice is called the **trivial representation**.

Weak congruence lattice representation problem 1

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

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Weak congruence lattice representation problem 2

Find a non-trivial representation of an algebraic lattice by a weak congruence lattice of an algebra.

Δ-suitable elements of a lattice

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Let L be an algebraic lattice. An element $a \in L$ is said to be **Δ-suitable** if there is an algebra $\mathcal{A}$ such that the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ is isomorphic to L, and Δ corresponds to a under the isomorphism.

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θ_a - a congruence on L which is the kernel of μ;
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$M_a := \{\overline{x} \mid x \in L\}$.

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- *if $x \wedge y \neq \mathbf{0}$ then $\overline{x \vee y} = \overline{x} \vee \overline{y}$;*
- *if $x \neq \mathbf{0}$ and $\overline{x} < y$, then $\overline{y \wedge a} \neq y \wedge a$;*

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;
- *if* $x \neq \mathbf{0}$ *and* $\overline{x} < y$, *then* $\overline{y \wedge a} \neq y \wedge a$;
- *if* $x \prec a$, *then* $\bigvee(y \in {\uparrow}a \mid y \vee \overline{x} < \mathbf{1}) \neq \mathbf{1}$;

Proposition

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- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;
- *if* $x \neq \mathbf{0}$ *and* $\overline{x} < y$, *then* $\overline{y \wedge a} \neq y \wedge a$;
- *if* $x \prec a$, *then* $\bigvee(y \in \uparrow a \mid y \vee \overline{x} < \mathbf{1}) \neq \mathbf{1}$;
- *If* $y \in \downarrow a$ *and* $x \prec y$, *then there exists* $z \in [y, \overline{y}]$, *such that*
  - *for all* $t \in [x, \overline{x}]$, *the set* $\{c \in \mathrm{Ext}(t) \mid c \leqslant z\}$ *is either empty or has the top element, and*
  - *for all* $t \in [x, \overline{x}]$, *the set* $\{c \in \mathrm{Ext}(t) \mid c \nleqslant z\}$ *is an antichain (possibly empty), where*

$$\mathrm{Ext}(t) := \{w \in [y, \overline{y}] \mid w \cap \overline{x} = t\}.$$

We start from an algebra $\mathcal{A} = (A, F)$, and L a complete lattice. **Fuzzy weak congruence** is a fuzzy relation R on A which is symmetric, transitive, compatible and weakly reflexive: for every $c \in F$, a nullary operation, $R(c, c) = 1$.

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Fuzzy weak congruence is a fuzzy relation R on A which is symmetric, transitive, compatible and weakly reflexive: for every $c \in F$, a nullary operation, $R(c, c) = 1$.
It is easy to check that all cut relations are weak congruences.

Again we start from an algebra $\mathcal{A} = (A, F)$, and L a complete lattice.
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HOWEVER
It is possible to represent a wider class of lattices by fuzzy weak congruence lattices than by classical weak congruence lattices.

Simple example: 2 element groupoid $(\{a, b\}, *)$ with two subgroupoids $\{a\}$ and $\{b\}$. Let L be 4 element Boolean lattice. Then the fuzzy weak congruence lattice has 25 elements, and it is not possible to represent this lattice by a weak congruence lattice.

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Thank you for your attention!