A strenthennigs of the Poincaré recurrence theorem on MV-algebras

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The strong version of the Poincaré recurrence theorem

$(\Omega, \mathcal{S}, P)$ - probability space
$T : \Omega \to \Omega$
P-measure preserving transformation

$$\forall A \in \mathcal{S} : P(T^{-1}(A) = P(A)$$

Theorem (Poincaré)

For any $A \in \mathcal{S}$ almost every point of A returns to A infinitely many times.

an l-group

an algebraic structure $(G, +, \leq)$,

$(G, +)$ is a commutative group,

$(G, \leq)$ is a lattice

$$a \leq b \Longrightarrow a + c \leq b + c$$

MV-algebra

an algebraic structure $(M, 0, u, \neg, \oplus, \odot)$,

0 is the neutral element in G

u is a positive element

$$M = \{x \in G; 0 \leq x \leq u\}$$

$\neg : M \to M$ is a unary operation

$$\neg x = u - x$$

$\oplus, \odot$ two binary operations

$$a \oplus b = (a + b) \wedge u$$

$$a \odot b = (a + b - u) \vee 0$$

Example

$\mathcal{S}$ - an algebra of subsets of a set Ω

Then $\mathcal{S}$ is an MV-algebra

If we identify sets A with their characteristic functions

l-group $(G, +, \leq)$ consists of all measurable functions

$+$ is the sum of functions

$\leq$ corresponds to the set inclusion

$$0 = 0_\Omega, u = 1_\Omega,$$

$$\neg\chi_A = \chi_{A^c} = 1_\Omega - \chi_A,$$

$$\chi_A \oplus \chi_B = \chi_{A \cup B},$$

$$\chi_A \odot \chi_B = \chi_{A \cap B}.$$

MV-algebra with product

MV-algebra with a commutative and associative binary operation $\star$ satisfying the following conditions

(i) $a \star u = a$,

(ii) $a * (b \oplus c) = (a \star b) \oplus (a \star c)$,

(iii) $a_n \nearrow a \Longrightarrow a_n \star b \nearrow a \star b$.

State on an MV-algebra M is a mapping $m : M \to [0,1]$ satisfying the following conditions:

(i) $m(u) = 1, m(0) = 0,$

(ii) $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b),$

(iii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a).$

Theorem

Let $(M, \star)$ *be a* σ*-complete MV-algebra with product. Let* $m : M \to [0,1]$ *satisfy the following conditions:*

1. $m(0) = 0$,
2. $a \le b \Longrightarrow m(a) \le m(b)$,
3. $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$.

Let $\tau : M \to M$ *satisfy the conditions*

4. $\tau(0) = 0$,
5. $a \le b \Longrightarrow \tau(a) \le \tau(b)$,
6. $\tau(a \odot b) = \tau(a) \odot \tau(b)$,
7. $m(\tau(a)) = m(a)$ *for all* $a \in M$.

Then for any $a \in M$ *and any* $k \in N$ *there holds*

$$m(a \star \Pi_{i=k}^{\infty} \tau(\neg a)) = 0.$$

$$m(a \star \Pi_{i=k}^{\infty} \tau(\neg a)) = 0.$$

$$P(\{x \in A; \forall i \geq k, T^i(x) \notin A\}) = 0$$

$$\forall k : P(\{x \in A; \exists i \geq k, T^i(x) \in A\}) = 1$$

$$P(\bigcap_{k=1}^{\infty} \{x \in A; \exists i \geq k, T^i(x) \in A\}) = 1$$

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Definition

Let $\mathcal{M}$ be an MV-algebra with product. A subset $\mathcal{N} \subset \mathcal{M}$ is called **ideal** if is satisfies the following conditions:

- $0 \in \mathcal{N}$.
- If $a \leq b, a \in \mathcal{M}, b \in \mathcal{N}$, then $a \in \mathcal{N}$.

A mapping $\tau : \mathcal{M} \to \mathcal{M}$ is called **conservative** if the following conditions hold:

- If $(\tau^i(b))_{i=0}^{\infty}$ forms a disjoint system (i.e. $\tau^i(b) \odot \tau^j(b) = 0$ for $i \neq j$) then $b = 0$.
- $\tau(a \odot b) = \tau(a) \odot \tau(b)$ for any $a, b \in \mathcal{M}$.
- $a \leq b$ implies $\tau(a) \leq \tau(b)$.
- $b \in \mathcal{N} \iff \tau(b) \in \mathcal{N}$.

Theorem

Let $\mathcal{M}$ be a σ-complete MV-algebra with product, $\mathcal{N} \subset \mathcal{M}$ its ideal, $\tau : \mathcal{M} \to \mathcal{M}$ be a conservative mapping. Then

$$a \star \Pi_{i=k}^{\infty} \tau^i(\neg a) \in \mathcal{N}$$

for any $a \in \mathcal{M}$ and any $k \in N$.