An aggregation operator designed for solving bilevel linear programming

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Bilevel linear programming problem (BLPP)

P^U – upper level problem,
$P^L = (P_1^L, P_2^L, ..., P_n^L)$ – lower level problems:

$$P^U: \quad y_0(x) = c_{01}x_1 + c_{02}x_2 + ... + c_{0k}x_k \longrightarrow \min$$

$$P_1^L: \quad y_1(x) = c_{11}x_1 + c_{12}x_2 + ... + c_{1k}x_k \longrightarrow \min$$

$$\cdots$$

$$P_n^L: \quad y_n(x) = c_{n1}x_1 + c_{n2}x_2 + ... + c_{nk}x_k \longrightarrow \min$$

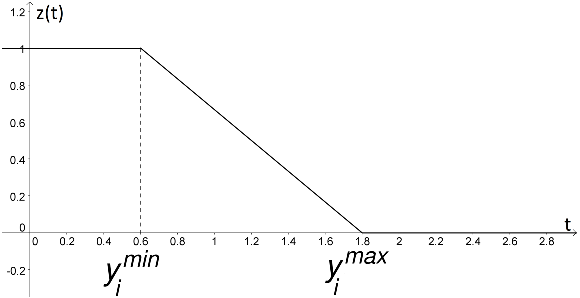
$$D: \begin{cases} a_{j1}x_1 + a_{j2}x_2 + ... + a_{jk}x_k \le b_j, \ j = 1, ..., m, \\ x_l \ge 0, \ l = 1, ..., k. \end{cases}$$

We suppose that D is non-empty bounded set.

Membership functions of objectives

$$y_i^{min} = \min_{x \in D} y_i(x), \quad y_i^{max} = \max_{x \in D} y_i(x), \quad i = 0, \ldots, n.$$

$$z_i(y_i(x)) = \begin{cases} 1, & y_i(x) < y_i^{min}, \\ \dfrac{y_i(x) - y_i^{max}}{y_i^{min} - y_i^{max}}, & y_i^{min} \le y_i(x) \le y_i^{max}, \\ 0, & y_i(x) > y_i^{max}. \end{cases}$$



Method for solving multi-objective linear programming problem

We denote:

$$\mu_i(x) = z_i(y_i(x)), i = \overline{0, n}.$$

Using the notation above it is reasonable to rewrite the problem:

$$\min_{i \in \{0,\dots,n\}} \mu_i(x) \longrightarrow \max_{x \in D},$$

which can be reduced to the linear programming problem (H.J. Zimmermann, 1978):

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu_0(x) \geq \sigma \\ \dots \\ \mu_n(x) \geq \sigma \\ x \in D \end{cases}$$

Let us denote by (x^*, σ^*) the optimal solution.

Parameters of the BLPP solving algorithm

Parameters $\delta, \Delta_L, \Delta_U$ are introduced to find the solution x^{**} for BLPP (M. Sakawa, I. Nishizaki, 2002):

1. $\mu_0(x^{**}) \geq \delta$;
2. $\Delta_L \leq \Delta = \dfrac{\min\{\mu_1(x^{**}), ..., \mu_n(x^{**})\}}{\mu_0(x^{**})} \leq \Delta_U.$

$$\min_{i=\overline{1,n}} \mu_i(x) \longrightarrow \max_{x \in D,\ \mu_0(x) \geq \delta}$$

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu_0(x) \geq \delta \\ \mu_1(x) \geq \sigma \\ \dots \\ \mu_n(x) \geq \sigma \\ x \in D \end{cases}$$

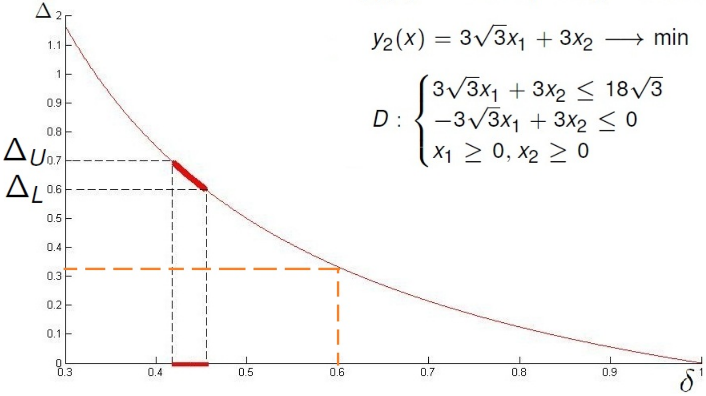
Parameters of the BLPP solving algorithm

$$y_0(x) = -x_2 \longrightarrow \min$$

$$y_1(x) = -3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$y_2(x) = 3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$D: \begin{cases} 3\sqrt{3}x_1 + 3x_2 \leq 18\sqrt{3} \\ -3\sqrt{3}x_1 + 3x_2 \leq 0 \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$



An aggregation operator designed for solving BLPP

To study in details the parameters of BLPP solving algorithm, a special aggregation has been constructed. The aggregation observes objective functions on the lower level considering the classes of equivalence generated by a function on the upper level.

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2, ..., \mu_n)(x) = \max_{\mu_0(x)=\mu_0(u)} \min(\mu_1(u), \mu_2(u), ..., \mu_n(u)),$$

where

$$\mu_0, \mu_1, ..., \mu_n \in [0,1]^D, \; x, u \in D.$$

General aggregation operator

Let $\mu_1, \mu_2, ..., \mu_n \in [0,1]^X$ be fuzzy sets and $\tilde{0}$, $\tilde{1}$ are indicators of $\varnothing$ and X respectively.

Definition

(A. Takaci, 2003) A mapping $\tilde{A}\colon \bigcup_n ([0,1]^X)^n \to [0,1]^X$ is called a general aggregation operator if the following conditions hold:

($\tilde{A}1$) $\tilde{A}(\tilde{0}, \ldots, \tilde{0}) = \tilde{0}$;

($\tilde{A}2$) $\tilde{A}(\tilde{1}, \ldots, \tilde{1}) = \tilde{1}$;

($\tilde{A}3$) $\forall \mu_1, \mu_2, ..., \mu_n, \eta_1, \eta_2, ..., \eta_n \in [0,1]^X$:
$\{\mu_i \preceq \eta_i,\ i = \overline{1,n}\} \Longrightarrow \{\tilde{A}(\mu_1, \ldots, \mu_n) \preceq \tilde{A}(\eta_1, \ldots, \eta_n)\}$.

General aggregation operator

- T-extension of an aggregation operator A

$$\tilde{A}(\mu_1, ..., \mu_n)(x) = \sup_{x=A(x_1,...,x_n)} T(\mu_1(x_1), ..., \mu_n(x_n))$$

- pointwise extension of an aggregation operator A

$$\tilde{A}(\mu_1, ..., \mu_n)(x) = A(\mu_1(x), ..., \mu_n(x))$$

Here

$$\mu_1, ..., \mu_n \in [0, 1]^X, \; x, x_1, ...x_n \in X.$$

Factoraggregation

Definition

General aggregation operator $\tilde{A}_{\mu_0}$ is called a *factoraggregation* of fuzzy sets $\mu_1, \mu_2, \ldots, \mu_n$ by means of fuzzy set μ_0 if

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2, \ldots, \mu_n)(x) = \sup_{\mu_0(u)=\mu_0(x)} A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u))$$

where

$$\mu_0, ..., \mu_n \in [0,1]^X, \ x, u \in X.$$

Equivalence relation generated by μ_0:

$$x \sim_{\mu_0} y \Longleftrightarrow \mu_0(x) = \mu_0(y).$$

Relation $\sim_{\mu_0}$ factorizes X into the classes of equivalence:

$$X^\alpha = \{x \in X | \mu_0(x) = \alpha\}.$$

Properties of factoraggregation

- Associativity

$$\tilde{A}_{\mu_0}(\mu_1, \tilde{A}_{\mu_0}(\mu_2, \mu_3)) = \tilde{A}_{\mu_0}(\tilde{A}_{\mu_0}(\mu_1, \mu_2), \mu_3)$$

  A is associative $\Longrightarrow$ $\tilde{A}_{\mu_0}$ is associative.

- Symmetry

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2) = \tilde{A}_{\mu_0}(\mu_2, \mu_1)$$

  A is symmetric $\Longrightarrow$ $\tilde{A}_{\mu_0}$ is symmetric.

- Idempotent elements
  An element μ is idempotent for $\tilde{A}_{\mu_0}$ if the following condition is satisfied:

$$x \sim_{\mu_0} y \Longrightarrow x \sim_{\mu} y.$$

Properties of factoraggregation

- An element a is an absorbing element of operator $A \Longrightarrow$
$\Longrightarrow \tilde{a}(x) \equiv a$ is an absorbing element of operator $\tilde{A}_{\mu_0}$:
$$\tilde{A}_{\mu_0}(\mu_1, \ldots, \mu_{i-1}, \tilde{a}, \mu_{i+1}, \ldots, \mu_n) = \tilde{a}.$$

- An element e is a neutral element of operator $A \Longrightarrow$
$\Longrightarrow \tilde{e}(x) \equiv e$ is a neutral element of operator $\tilde{A}_{\mu_0}$:
$$\tilde{A}_{\mu_0}(\mu_1, \ldots, \mu_{i-1}, \tilde{e}, \mu_{i+1}, \ldots, \mu_n) =$$
$$\tilde{A}_{\mu_0}(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n).$$

Factoraggregation applied for solving BLPP

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2, ..., \mu_n)(x) = \max_{\mu_0(x)=\mu_0(u)} \min(\mu_1(u), \mu_2(u), ..., \mu_n(u)),$$

where

$$\mu_1, ..., \mu_n \in [0, 1]^D, \; x, u \in D.$$

Properties:

- symmetry,
- associativity,
- existence of idempotent elements,
- $\tilde{0}$ is absorbing element,
- $\tilde{1}$ is neutral element.

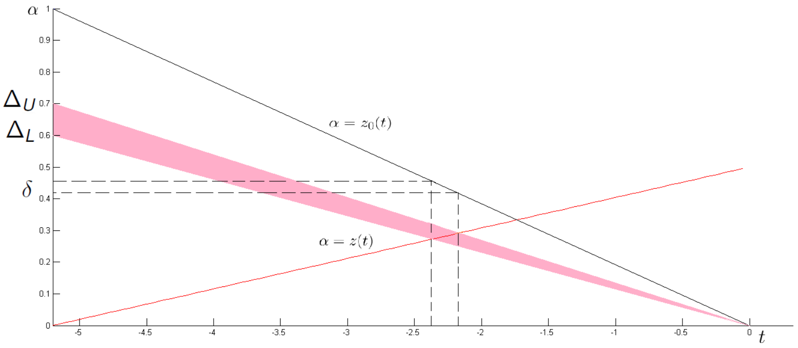
Denoting

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2, ..., \mu_n)(x) = \mu(x),$$

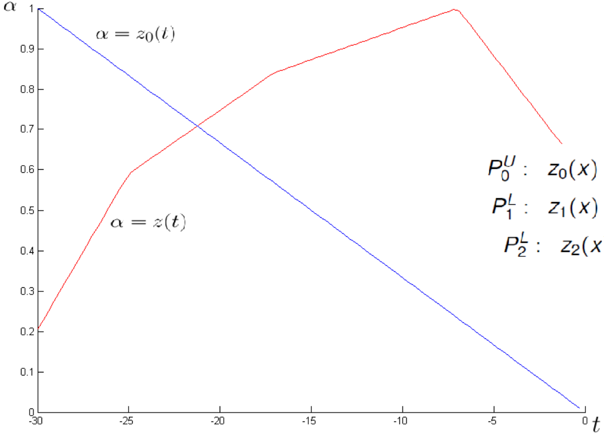
$$z(y_0(x)) = \mu(x),$$

$$t = y_0(x),$$

we consider the figures $\alpha = z_0(t)$ *and* $\alpha = z(t)$:



Analysis of BLPP parameters



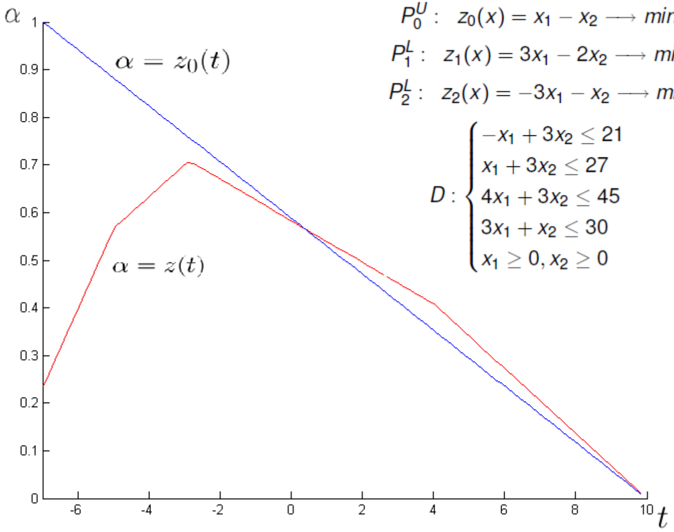
$$P_0^U: \quad z_0(x) = -3x_1 - x_2 \longrightarrow min$$

$$P_1^L: \quad z_1(x) = 3x_1 - 2x_2 \longrightarrow min$$

$$P_2^L: \quad z_2(x) = x_1 - x_2 \longrightarrow min$$

$$D: \begin{cases} -x_1 + 3x_2 \leq 21 \\ x_1 + 3x_2 \leq 27 \\ 4x_1 + 3x_2 \leq 45 \\ 3x_1 + x_2 \leq 30 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Analysis of BLPP parameters



$$P_0^U: \quad z_0(x) = x_1 - x_2 \longrightarrow min$$

$$P_1^L: \quad z_1(x) = 3x_1 - 2x_2 \longrightarrow min$$

$$P_2^L: \quad z_2(x) = -3x_1 - x_2 \longrightarrow min$$

$$D: \begin{cases} -x_1 + 3x_2 \leq 21 \\ x_1 + 3x_2 \leq 27 \\ 4x_1 + 3x_2 \leq 45 \\ 3x_1 + x_2 \leq 30 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

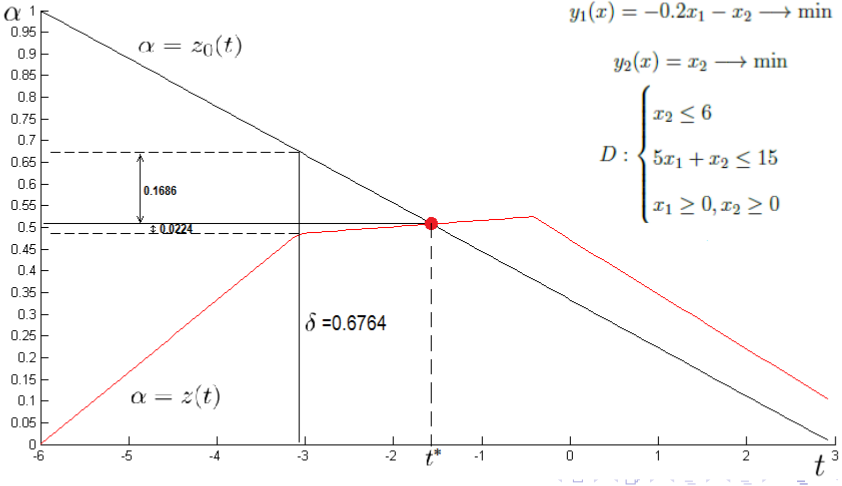
Properties of function $z(t)$ obtained by the factoraggregation

- $\sigma^* = min\{z(t^*),\ z_0(t^*)\},\ where\ t^* = y_0(x^*).$
- $\max\limits_{t \in [y_0^{min},\ t^*]} z(t) = z(t^*).$
- Function z is monotone on interval $[y_0^{min},\ t^*]$:

$$\forall t^1,\ t^2 \in [y_0^{min},\ t^*] : t^1 < t^2 \Longrightarrow z(t^1) \leq z(t^2).$$

- Function z is convex on interval $[y_0^{min}, t^*]$.

Analysis of BLPP parameters



$$y_0(x) = x_1 - x_2 \longrightarrow \min$$

$$y_1(x) = -0.2x_1 - x_2 \longrightarrow \min$$

$$y_2(x) = x_2 \longrightarrow \min$$

$$D : \begin{cases} x_2 \leq 6 \\ 5x_1 + x_2 \leq 15 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Thank you for your attention!