Musings about Triple Representation Theorem for effect algebras

Josef Niederle and Jan Paseka
Department of Mathematics and Statistics
Masaryk University, Brno, Czech Republic
niederle,paseka@math.muni.cz

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Joint work of J. Niederle and J. Paseka

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We present a solution of G. Jenča's problem which homogeneous effect algebras E may be represented via the set $S(E)$ of sharp elements, the set $M(E)$ of meager elements and order relation between meager and sharp elements.

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Basic definition – effect algebras

Definition (D. Foulis and M.K. Bennett, 1994)

A partial algebra $(E;\oplus,0,1)$ is called an **effect algebra** if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x,y,z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

Example

Let $E = [0,1] \subseteq \mathbb{R}$. We put $x \oplus y = x + y$ iff $x + y \leq 1$. Hence $\frac{3}{4} \oplus \frac{4}{5}$ does not exist in E.

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A subset $Q \subseteq E$ is called a *sub-effect algebra* of E if

(i) $1 \in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

An effect algebra E is called an *orthoalgebra* if $x \oplus x$ exists implies that $x = 0$.

On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y\,.$$

If E with the defined partial order is a (complete) lattice then $(E; \oplus, 0, 1)$ is called a *(complete) lattice effect algebra.*

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Sharp and meager elements

An element w of an effect algebra E is called *sharp* if $w \wedge w' = 0$.

The well known fact is that in every lattice effect algebra E the subset $\mathrm{S}(E) = \{w \in E \mid w \wedge w' = 0\}$ is a sub-lattice effect algebra of E being an orthomodular lattice.

In what follows set

$$\mathrm{M}(E) = \{x \in E \mid \text{ if } v \in \mathrm{S}(E) \text{ satisfies } v \leq x \text{ then } v = 0\}.$$

An element $x \in \mathrm{M}(E)$ is called *meager*.

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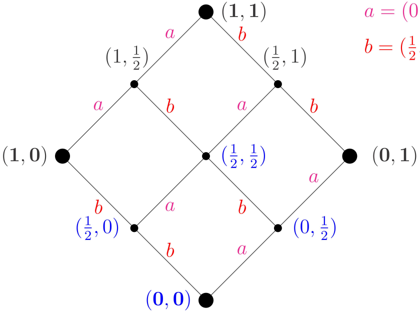

$a = (0, \frac{1}{2})$

$b = (\frac{1}{2}, 0)$

$\oplus$	$(\mathbf{0},\mathbf{0})$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$
$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{0})$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$
$(\mathbf{0},\mathbf{1})$	$(\mathbf{0},\mathbf{1})$	$\not\perp$	$(\frac{1}{2},1)$	$\not\perp$
$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{0})$	$(1,\frac{1}{2})$	$\not\perp$	$\not\perp$
$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	$\not\perp$	$\not\perp$	$\not\perp$

$$\begin{aligned} \mathrm{S}(E) &= \{(\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{0}),(\mathbf{1},\mathbf{1})\} \\ \mathrm{M}(E) &= \{(0,0),(0,\tfrac{1}{2}),(\tfrac{1}{2},0),(\tfrac{1}{2},\tfrac{1}{2})\} \end{aligned}$$

Sharp dominancy

Definition

(Gudder - 1998) An effect algebra E is called ***sharply dominating*** if for every $x \in E$ there exists $\widehat{x} \in S(E)$ such that

$$\widehat{x} = \bigwedge_E \{w \in S(E) \mid x \leq w\} = \bigwedge_{S(E)} \{w \in S(E) \mid x \leq w\}.$$

Note that clearly E is sharply dominating iff for every $x \in E$ there exists $\widetilde{x} \in S(E)$ such that

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Proposition (G. Jenča)

Let E be a sharply dominating effect algebra. Then every $x \in E$ has a unique decomposition $x = x_S \oplus x_M$, where $x_S \in S(E)$ and $x_M \in M(E)$, namely $x = \widetilde{x} \oplus (x \ominus \widetilde{x})$.

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Prototypical example for effect algebras

Example

The most important example of an effect algebra for quantum mechanical investigations is a Hilbert space effect algebra. Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{E}(\mathscr{H})$ be the set of linear operators on $\mathscr{H}$ that satisfy $0 \leq A \leq I$.

That is $0 \leq \langle Ax, x\rangle \leq \langle Ix, x\rangle$ for all $x \in \mathscr{H}$. For $A, B \in \mathscr{E}(\mathscr{H})$ we write $A \perp B$ if $A + B \in \mathscr{E}(\mathscr{H})$ and in this case we define $A \oplus B = A + B$. If we define $A' = I - A$ for $A \in \mathscr{E}(\mathscr{H})$, it is clear that $(\mathscr{E}(\mathscr{H}); \oplus, 0, I)$ is an effect algebra.

Denoting the set of projections on $\mathscr{H}$ by $\mathscr{P}(\mathscr{H})$ we have $\mathscr{P}(\mathscr{H}) \subseteq \mathscr{E}(\mathscr{H})$ and it is clear that $\mathscr{P}(\mathscr{H})$ is an orthoalgebra in $\mathscr{E}(\mathscr{H})$ such that $\mathrm{S}(\mathscr{E}(\mathscr{H})) = \mathscr{P}(\mathscr{H})$.

Also, $\mathscr{E}(\mathscr{H})$ is sharply dominating and $\widehat{A}$ is the projection onto the closure of the range of A for $A \in \mathscr{E}(\mathscr{H})$. Hence $\mathrm{M}(\mathscr{E}(\mathscr{H})) = \{A \in \mathscr{E}(\mathscr{H}) \mid cl(\mathrm{Im}(I - A)) = \mathscr{H}\}$.

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Generalized effect algebras

Definition

(1) A *generalized effect algebra* $(E; \oplus, 0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying, for any $x, y, z \in E$, conditions

- (GE1) $x \oplus y = y \oplus x$ if one side is defined,
- (GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,
- (GE3) if $x \oplus y = x \oplus z$ then $y = z$,
- (GE4) if $x \oplus y = 0$ then $x = y = 0$,
- (GE5) $x \oplus 0 = x$ for all $x \in E$.

(2) A binary relation $\leq$ (being a partial order) and a partial binary operation $\ominus$ on E can be defined by:

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- (GE1) $x \oplus y = y \oplus x$ if one side is defined,
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(2) A binary relation $\leq$ (being a partial order) and a partial binary operation $\ominus$ on E can be defined by:

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Canonical example from quantum theory for modelling unsharp measurements in a Hilbert space

Example (S. Pulmannová, Z. Riečanová and M. Zajac)

Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space and

$$\mathscr{V}(\mathscr{H}) = \{A : D(A) \to \mathscr{H} \mid A \geq 0, \overline{D(A)} = \mathscr{H} \text{ and } D(A) = \mathscr{H} \text{ if } A \text{ is bounded}\}$$

Let $\oplus_{\mathscr{D}}$ be defined for $A, B \in \mathscr{V}(\mathscr{H})$ by $A \oplus_{\mathscr{D}} B = A + B$ (the usual sum) iff

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Outline

1. Introduction
2. Basic definitions
3. Triple Representation Theorem for complete effect algebras
4. Triple Representation Theorem - general version of Jenča's problem
5. The solution of Jenča's problem
6. Main theorem

Triple Representation Theorem for complete effect algebras

In what follows E will be always a complete effect algebra. Then $\mathrm{S}(E)$ is a sub-effect algebra of E and $\mathrm{M}(E)$ equipped with a partial operation $\oplus_{\mathrm{M}(E)}$ which is defined, for all $x, y \in \mathrm{M}(E)$, by $x \oplus_{\mathrm{M}(E)} y$ exists if and only if $x \oplus_E y$ exists and $x \oplus_E y \in \mathrm{M}(E)$ in which case $x \oplus_{\mathrm{M}(E)} y = x \oplus_E y$ is a generalized effect algebra. Moreover, we have a map $h : \mathrm{S}(E) \to 2^{\mathrm{M}(E)}$ that is given by $h(s) = \{x \in \mathrm{M}(E) \mid x \leq s\}$.

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Triple Representation Theorem for complete effect algebras

In other words, $\oplus = \boxplus$ whenever $(E; \oplus, 0_\oplus, 1_\oplus)$ with induced order $\leq$ and $(E; \boxplus, 0_\boxplus, 1_\boxplus)$ with induced order $\sqsubseteq$ are complete lattice effect algebras and

$$\begin{aligned}
&\mathrm{M}_\oplus(E) = \mathrm{M}_\boxplus(E) = \mathrm{M}(E)\\
&\mathrm{S}_\oplus(E) = \mathrm{S}_\boxplus(E) = \mathrm{S}(E)\\
&\oplus_{\mathrm{M}(E)} = \boxplus_{\mathrm{M}(E)}\\
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&\leq \cap(\mathrm{M}(E) \times \mathrm{S}(E)) = \sqsubseteq \cap(\mathrm{M}(E) \times \mathrm{S}(E))\\
&0_\oplus = 0_\boxplus\\
&1_\oplus = 1_\boxplus.
\end{aligned} \tag{T}$$

Outline

Homogeneous effect algebras

An effect algebra E satisfies the *Riesz decomposition property* (or RDP) if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$, there are u_1, u_2 such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

(i) Every lattice effect algebra with RDP can be organized into an MV-algebra and conversely.
(ii) Every MV-algebra which is an orthoalgebra is a Boolean algebra.

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Blocks of homogeneous effect algebras

Let E be a homogeneous effect algebra. We say that a subset B of E is called a *block* of E if it is a maximal sub-effect algebra of E with the Riesz decomposition property.

Let E be a homogeneous effect algebra. Then

(i) Every homogeneous effect algebra is a union of its blocks.

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Archimedeanicity

An effect algebra E is *Archimedean* if for all $x \in E$, $x \neq 0$ there exists positive integer $n_x = max\{n \in \mathbb{N} \mid nx = \underbrace{x \oplus x \oplus \cdots \oplus x}_{n\text{-times}} \text{ exists}\}$.

Formulation of of Jenča's problem

Find the largest subclass of sharply dominating Archimedean homogeneous effect algebras E such that the triple $((\mathrm{S}(E), \oplus), (\mathrm{M}(E), \oplus), h)$ characterizes E up to isomorphism within this class.

Outline

Triple Representation Theorem for homogeneous effect algebras

Jenča proved the Triple Representation Theorem for complete lattice effect algebras, we for sharply dominating atomic Archimedean lattice effect algebras and for sharply dominating orthocomplete homogeneous effect algebras. These classes share the following properties.

being sharply dominating
being Archimedean
homogeneity
$\mathrm{M}(E)$ being a meet-semilattice
$z \in B \implies [\widetilde{z}, z] \subseteq B$ for every block B
$(\forall y, z \in \mathrm{M}(E))\, \widehat{z} = \widehat{y} \implies \widehat{y \wedge z} = \widehat{y}$

(C)

Triple Representation Theorem *The triple $((\mathrm{S}(E), \oplus), (\mathrm{M}(E), \oplus), h)$ characterizes E up to isomorphism within the class of all effect algebras satisfying (C).*

We have to construct an isomorphic copy of the original effect algebra E from the triple $(\mathrm{S}(E), \mathrm{M}(E), h)$. To do this we will first construct the following mappings in terms of the triple.

(M1) The mapping $\widehat{\ } : \mathrm{M}(E) \to \mathrm{S}(E)$.

(M2) For every $s \in \mathrm{S}(E)$, there is a partial mapping $\pi_s : \mathrm{M}(E) \to h(s)$, which is given by $\pi_s(x) = x \wedge_E s$ whenever $\pi_s(x)$ is defined.

(M3) The mapping $R : \mathrm{M}(E) \to \mathrm{M}(E)$ given by $R(x) = \widehat{x} \ominus_E x$.

(M4) The partial mapping $S : \mathrm{M}(E) \times \mathrm{M}(E) \to \mathrm{S}(E)$ given by $S(x,y)$ is defined if and only if the set $\mathcal{S}(x,y) = \{z \in \mathrm{S}(E) \mid z = (z \wedge x) \oplus_E (z \wedge y)\}$ has a top element $z_0 \in \mathcal{S}(x,y)$ in which case $S(x,y) = z_0$.

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We have to construct an isomorphic copy of the original effect algebra E from the triple $(\mathrm{S}(E),\mathrm{M}(E),h)$. To do this we will first construct the following mappings in terms of the triple.

(M1) The mapping $\widehat{\ }:\mathrm{M}(E)\to\mathrm{S}(E)$.

(M2) For every $s\in\mathrm{S}(E)$, there is a partial mapping $\pi_s:\mathrm{M}(E)\to h(s)$, which is given by $\pi_s(x)=x\wedge_E s$ whenever $\pi_s(x)$ is defined.

(M3) The mapping $R:\mathrm{M}(E)\to\mathrm{M}(E)$ given by $R(x)=\widehat{x}\ominus_E x$.

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The sum of meager elements

Lemma

Let E be an effect algebra satisfying (C), $x,y \in \mathrm{M}(E)$. Then $x \oplus_E y$ exists in E iff $S(x,y)$ is defined in terms of the triple $(\mathrm{S}(E), \mathrm{M}(E), h)$ and $(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)) \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y))$ exists in $\mathrm{M}(E)$ such that $(x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)) \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)) \in h(S(x,y)')$. Moreover, in that case

$$x \oplus_E y = \underbrace{S(x,y)}_{\in \mathrm{S}(E)} \oplus_E \underbrace{((x \ominus_{\mathrm{M}(E)} (S(x,y) \wedge x)) \oplus_{\mathrm{M}(E)} (y \ominus_{\mathrm{M}(E)} (S(x,y) \wedge y)))}_{\in \mathrm{M}(E)}.$$

Outline

Main theorem

Theorem

Let E be an effect algebra satisfying (C). Let $\mathrm{T}(E)$ *be a subset of* $\mathrm{S}(E) \times \mathrm{M}(E)$ *given by*

$$\mathrm{T}(E) = \{(z_S, z_M) \in \mathrm{S}(E) \times \mathrm{M}(E) \mid z_M \in h(z_S')\}.$$

Equip $\mathrm{T}(E)$ *with a partial binary operation* $\oplus_{\mathrm{T}(E)}$ *with* $(x_S, x_M) \oplus_{\mathrm{T}(E)} (y_S, y_M)$ *is defined if and only if*

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In this case $(z_S, z_M) = (x_S, x_M) \oplus_{\mathrm{T}(E)} (y_S, y_M)$. Let $0_{\mathrm{T}(E)} = (0_E, 0_E)$ and $1_{\mathrm{T}(E)} = (1_E, 0_E)$. Then $\mathrm{T}(E) = (\mathrm{T}(E), \oplus_{\mathrm{T}(E)}, 0_{\mathrm{T}(E)}, 1_{\mathrm{T}(E)})$ is an effect algebra and the mapping $\varphi : E \to \mathrm{T}(E)$ given by $\varphi(x) = (\widetilde{x}, x \ominus_E \widetilde{x})$ is an isomorphism of effect algebras.

Problem

By a result of Pulmannová $\mathscr{E}(\mathscr{H})$ can be covered by MV-algebras. But $\mathscr{E}(\mathscr{H})$ is not homogeneous unless $\dim(\mathscr{H}) = 1$.

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Thank you for your attention.