Analysis of generalized square of opposition with intermediate quantifiers

Petra Murinová

Centre of Excellence IT4Innovations - Division University of Ostrava Institute for Research and Applications of Fuzzy Modeling
University of Ostrava
Czech Republic
petra.murinova@osu.cz

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Outline

Motivation

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- Elaboration of theory of intermediate quantifiers from Peterson's book Intermediate Quantifiers - analysis of complete square of opposition with the quantifiers *almost all, most, many*, etc.
- In the book of Peterson is no formal mathematical system.
- Application of Łukasewicz fuzzy type theory.

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Contradictory, Contrary and Subcontrary

Contradictory

- x and y are contradictories iff x and y cannot both be true; x and y cannot both be false

Contraries

- x and y are contraries iff x and y cannot both be true; x and y can both be false

Sub-contraries

- x and y are sub-contraries iff x and y cannot both be false; x and y can both be true

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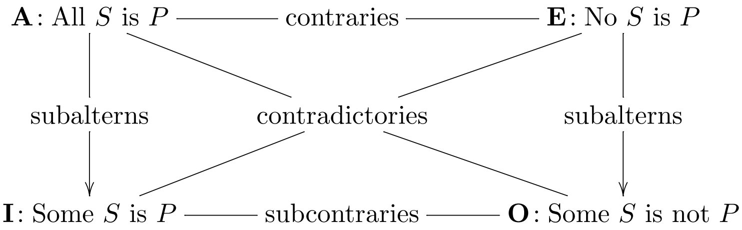
Aristotle's square



Complete square of opposition

- The first version of the complete square of opposition was introduced by P. Peterson in (1979) with "Almost-all" and "Many".
- Thompson extends the approach by the intermediate quantifier "Most" and introduced a complete square of opposition with contradictions, contraries and subalterns as follows:

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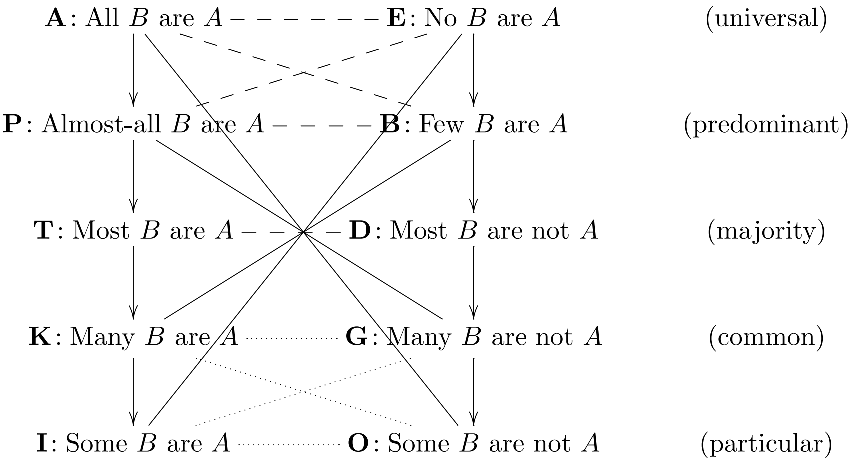

Structure of truth values-MV$_\Delta$-algebra

MV$_\Delta$-algebra

$$\mathcal{L}. = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta \rangle, \quad (1)$$

1. $\langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \rangle$ is an MV-algebra with involutive negation,

where

- $\Delta a \vee \neg\Delta a = 1$,
- $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,
- $\Delta a \leq a, \qquad \Delta a \leq \Delta\Delta a$,
- $\Delta(a \rightarrow b) \leq \Delta a \rightarrow \Delta b$,
- $\Delta\mathbf{1} = \mathbf{1}$.

Example of MV_Δ-algebra

Standard Łukasiewicz algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1, \Delta \rangle \tag{2}$$

1. $\vee = \max$
2. $\wedge = \min$
3. $a \otimes b = \max(0, a + b - 1)$
4. $a \rightarrow b = 1 \wedge (1 - a + b)$
5. $\neg a = a \rightarrow 0 = 1 - a$

Basic syntactical elements

The language of Ł-FTTdenoted by J consists of:

- variables $x_\alpha, \ldots$
- special constants $c_\alpha, \ldots$ ($\alpha \in$ *Types*)
- λ and brackets
- $\mathbf{E}_{(o\alpha)\alpha}$ for every $\alpha \in$ *Types* for fuzzy equality,
- $\mathbf{C}_{(oo)o}$ for conjunction,
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Basic definitions

1. Equivalence: $\equiv \; := \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha, \qquad \alpha \in Types.$
2. Conjunction: $\wedge \; := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o}\, y_o) x_o.$
3. Delta connective: $\boldsymbol{\Delta} \; := \lambda x_o \mathbf{D}_{oo} x_o.$

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Derived connectives

1. Representation of truth: $\top := \lambda x_o x_o \equiv \lambda x_o x_o$.
2. Representation of falsity: $\bot := \lambda x_o x_o \equiv \lambda x_o \top$.
3. Negation: $\neg := \lambda x_o (x_o \equiv \bot)$.
4. Implication: $\Rightarrow \; := \lambda x_o \lambda y_o \, (x_o \wedge y_o) \equiv x_o$
5. $\&, \nabla, \vee$ are defined as in Łukasiewicz logic.
6. General quantifier: $(\forall x_\alpha) A_o := (\lambda x_\alpha A_o \equiv \lambda x_\alpha \top)$,
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Axioms and inference rules in Ł-FTT

- 17 axioms
- two inference rules where the rules *modus ponens and generalization* are the rules derivative.

Semantics in Ł-FTT

- A *frame* is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

1. $(M_\alpha)_{\alpha \in Types}$ is a basic frame
2. $\mathcal{L}_\Delta$ is MV-algebra with $\mathbf{\Delta}$
3. $=_\alpha$ is a fuzzy equality on M_α.

- We say that a frame $\mathcal{M}$ is a *model* of a theory T if all axioms are true in the degree $\mathbf{1}$ in $\mathcal{M}$.

Trichotomous evaluative linguistic expressions

TEE

- are special expressions of natural language, e.g., *small, big, about fourteen, very short, more or less deep, not thick*.
- Linguistic hedge can be
  - *narrowing — extremely, significantly, very*
  - *widening — more or less, roughly, quite roughly, very roughly*
  - *empty hedge*
- We will work with expressions: extremely big, very big, not small.
- T^{Ev} has 11 axioms.

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Theory of intermediate quantifiers T^{IQ}

1. is a special theory of Ł-FTT extending the theory T^{Ev} of evaluative linguistic expressions
2. we consider a special formula μ of type $o(o\alpha)(o\alpha)$ such that values of the measure are taken from the set of truth values
3. μ has four axioms

Definition of intermediate generalized quantifiers

Definitions of intermediate generalized quantifiers of the form "Quantifier B's are A"

(a) $(Q^{\forall}_{Ev}\, x)(B, A) := (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\, \&(\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B)z)),$

(b) $(Q^{\exists}_{Ev}\, x)(B, A) := (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\, \&(\exists x)(zx \wedge Ax)) \wedge Ev((\mu B)z)).$

Definition of intermediate generalized quantifiers

Explanation of definition of IGQ

Each formula above consists of three parts:

$$\underbrace{(\exists z)((\mathbf{\Delta}(z \subseteq B)}_{\text{“the greatest” part of } B\text{'s}} \quad \& \quad \underbrace{(\forall x)(z\,x \Rightarrow Ax))}_{\text{each } z\text{'s has } A} \quad \wedge \quad \underbrace{Ev((\mu B)z))}_{\text{size of } z \text{ is evaluated by } Ev} \tag{3}$$

Definition of intermediate generalized quantifiers with presupposition

Interpretation of “Quantifier B’s are A” with presupposition

(a) $({}^{*}Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\, \&\, (\exists x)zx\, \&\, (\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B)z)),$

(b) $({}^{*}Q^{\exists}_{Ev}\, x)(B, A) := (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\, \&\, (\exists x)zx\, \&\, (\exists x)(zx \wedge Ax)) \wedge Ev((\mu B)z)).$

where only non-empty subsets of B are considered.

"All", "No", "Almost all", "Few", "Most"

A: All B are $A := Q^{\forall}_{Bi\Delta}(B, A) \equiv (\forall x)(Bx \Rightarrow Ax)$,

E: No B are $A := Q^{\forall}_{Bi\Delta}(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax)$,

P: Almost all B are $A := Q^{\forall}_{Bi\,Ex}(B, A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\,Ex)((\mu B)z))$,

B: Few B are A ($:=$ Almost all B are not A) $:= Q^{\forall}_{Bi\,Ex}(B, \neg A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\,Ex)((\mu B)z))$,

T: Most B are $A := Q^{\forall}_{Bi\,Ve}(B, A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\,Ve)((\mu B)z))$,

D: Most B are not $A := Q^{\forall}_{Bi\,Ve}(B, \neg A) \equiv$
$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\,Ve)((\mu B)z))$,

"Many", "Some"

K: Many B are $A := Q^{\forall}_{\neg(Sm\bar{\nu})}(B, A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\bar{\nu})((\mu B)z)),$$

G: Many B are not $A := Q^{\forall}_{\neg(Sm\bar{\nu})}(B, \neg A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\bar{\nu})((\mu B)z)),$$

I: Some B are $A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\exists x)(Bx \wedge Ax),$

O: Some B are not $A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax).$

Generalized definitions in Ł-FTT

Contraries

$P_1, P_2 \in Form_o$ are contraries in T^{IQ} if in every model $\mathcal{M} \models T^{IQ}$ the following is true:

$$\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = \mathcal{M}(\bot).$$

We can alternatively say that P_1 and P_2 are contraries if $T^{IQ} \vdash P_1 \,\&\, P_2 \equiv \bot$.

Generalized definitions in Ł-FTT

Sub-contraries

$P_1, P_2 \in Form_o$ are sub-contraries in T^{IQ} if in every model $\mathcal{M} \models T^{IQ}$ the following is true:

$$\mathcal{M}(P_1) \oplus \mathcal{M}(P_2) = \mathcal{M}(\top).$$

We can alternatively say that P_1 and P_2 are sub-contraries if $T^{IQ} \vdash P_1 \nabla P_2$.

Generalized definitions in Ł-FTT

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Generalized definitions in Ł-FTT

Contradictories

$P_1, P_2 \in Form_o$ are contradictories in T^{IQ} if in every model $\mathcal{M} \models T^{\mathrm{IQ}}$ the following two equalities hold:

- $\mathcal{M}(\boldsymbol{\Delta} P_1) \otimes \mathcal{M}(\boldsymbol{\Delta} P_2) = \mathcal{M}(\bot)$,
- $\mathcal{M}(\boldsymbol{\Delta} P_1) \oplus \mathcal{M}(\boldsymbol{\Delta} P_2) = \mathcal{M}(\top)$.

Alternatively we can say that P_1 and P_2 are contradictories, if both $T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta} P_1 \,\&\, \boldsymbol{\Delta} P_2 \equiv \bot$ as well as $T^{\mathrm{IQ}} \vdash \boldsymbol{\Delta} P_1 \,\nabla\, \boldsymbol{\Delta} P_2$.

Generalized definitions in Ł-FTT

Subaltern

We say that A is a subaltern of S in T^{IQ} if in every model $\mathcal{M} \models T^{\mathrm{IQ}}$ the inequality

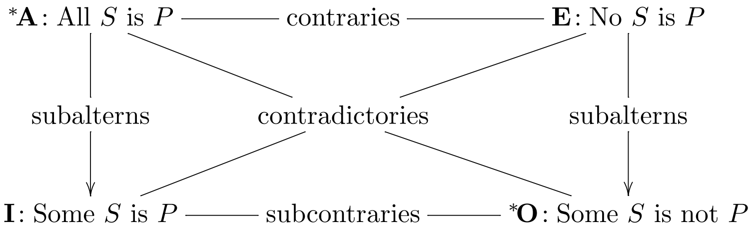
$$\mathcal{M}(A) \leq \mathcal{M}(S)$$

holds true. We will call S as *superaltern* of A. Alternatively we can say that A is a subaltern of S if $T^{\mathrm{IQ}} \vdash A \Rightarrow S$.

Properties of classical quantifiers in Ł-FTT

- The formulas ***A**, **E** are contraries in T^{IQ} ($T^{IQ} \vdash {}^{*}\mathbf{A} \,\&\, \mathbf{E} \equiv \bot$).
- If $T^{IQ} \vdash (\exists x)Bx$ then the formulas **A**, **E** are contraries in T^{IQ}.
- The formulas ***O** and **I** are sub-contraries in T^{IQ} ($T^{IQ} \vdash {}^{*}\mathbf{O} \nabla \mathbf{I}$).
- If $T^{IQ} \vdash (\exists x)Bx$, then the formulas **O** and **I** are sub-contraries in T^{IQ}.
- The formulas **A** and **O** are contradictories in T^{IQ}.
- The formulas **E** and **I** are contradictories in T^{IQ}.

Aristotle's square interpreted in T^{IQ}



Extension of the theory T^{IQ}

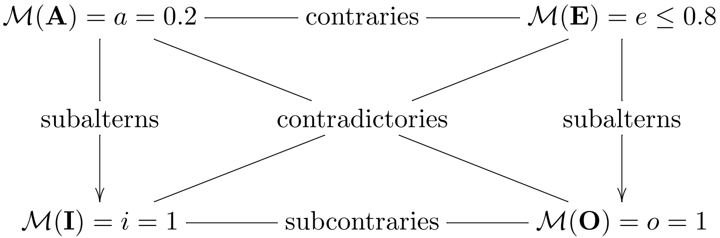
Theory $T[B, B']$

Let $B, B' \in Form_{o\alpha}$. The theory $T[B, B']$ is a consistent extension of T^{IQ} such that

(a) $T[B, B'] \vdash B \equiv B'$,

(b) $T[B, B'] \vdash (\exists x_\alpha)\boldsymbol{\Delta} Bx$ and $T[B, B'] \vdash (\exists x_\alpha)\boldsymbol{\Delta} B'x$.

Example of generalized Aristotelian square interpreted in $T[B, B']$



Properties of generalized quantifiers in $T[B, B']$

The main properties

- $T[B, B'] \vdash \mathbf{B} \,\&\, \mathbf{P} \equiv \bot$,
- $T[B, B'] \vdash \mathbf{D} \,\&\, \mathbf{T} \equiv \bot$,
- $T[B, B'] \vdash \mathbf{G} \,\&\, \mathbf{K} \equiv \bot$.
- $T[B, B'] \vdash \mathbf{G} \,\&\, \mathbf{P} \equiv \bot$,
- $T[B, B'] \vdash \mathbf{K} \,\&\, \mathbf{B} \equiv \bot$.

Properties of generalized quantifiers in $T[B, B']$

Derived properties

- $T[B, B'] \vdash \mathbf{E} \,\&\, \mathbf{K} \equiv \bot$,
- $T[B, B'] \vdash \mathbf{E} \,\&\, \mathbf{T} \equiv \bot$,
- $T[B, B'] \vdash \mathbf{E} \,\&\, \mathbf{P} \equiv \bot$,
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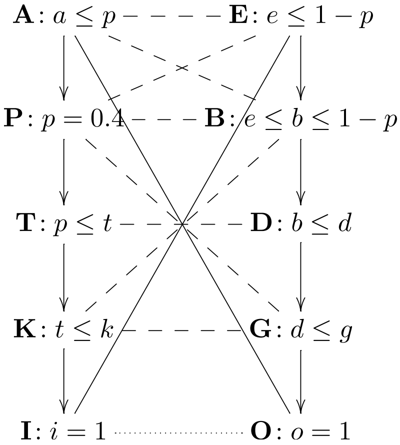
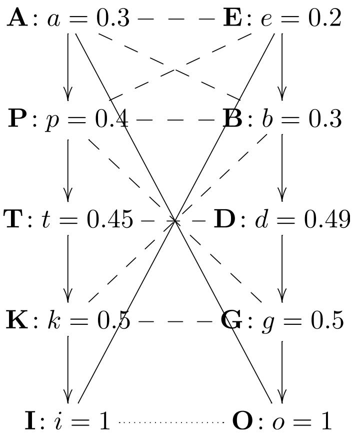
Example of generalized complete square



(such that $t \otimes d = 0$)

(such that $k \otimes g = 0$)

Generalized complete square of opposition



Main results

Results

- I developed Ł-FTT.
- I proposed generalized definitions of properties which characterize relations among intermediate generalized quantifiers in the generalized square of opposition.
- I formally proved validity of these relations.

Thank you for your attention.