Vertical Mixtures of Copulas

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2-Copula

Binary operation $C\colon [0,1]^2 \to [0,1]$

- 0 as an annihilator
- 1 as a neutral element
- for all $u_1 \leq u_2$ and $v_1 \leq v_2$ from $[0,1]$

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$$

Quasi-Copula

Binary operation $Q\colon [0,1]^2 \to [0,1]$

- nondecreasing in both operands
- 1 as a neutral element
- for all u_1, u_2, v_1, v_2 in $[0,1]$

$$|Q(u_1, v_1) - Q(u_2, v_2)| \leq |u_1 - u_2| + |v_1 - v_2|$$

Sklar Theorem

Theorem (Sklar)

A mapping $F_{XY}\colon [-\infty,\infty]^2 \to [0,1]$ is a joint distribution function of a random vector (X,Y) with marginal distributions F_X and F_Y respectively iff there exists a copula C_{XY} such that

$$F_{XY}(x,y) = C_{XY}\big(F_X(x), F_Y(y)\big)$$

holds for all $x, y \in [-\infty,\infty]$.

Corollary

Copulas are $[0,1]^2$-restrictions of probability distribution functions of random vectors with components distributed uniformly on $[0,1]$.

Copula and its Induced Measure

C-volume of a rectangle

Given a copula C and a rectangle $R = [u_1, u_2] \times [v_1, v_2]$ define C-volume of R by

$$V_C(R) = C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2).$$

For any two rectangles R_1, R_2 with a common edge if $R_1 \cup R_2$ is a rectangle again, then

$$V_C(R_1 \cup R_2) = V_C(R_1) + V_C(R_2).$$

C-measure

Given a copula C the induced C-measure is the completion of the σ-additive extension of V_C.

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Basic Facts

- every copula is a quasi-copula
- given two (quasi-)copulas A, B and $\alpha \in [0, 1]$ the operation $\alpha A + (1 - \alpha)B$ is a (quasi-)copula again
- quasi-copulas are nondecreasing in each variable
- quasi-copulas are continuous
- quasi-copulas admit first partial derivatives λ-almost everywhere

Prototypical examples

- $M(u, v) = \min\{u, v\}$
- $\Pi(u, v) = uv$
- $W(u, v) = \max\{u + v - 1, 0\}$

Outline

1. Vertical mixtures

Residual and Deresidual transform

Definition

For a binary operation $O\colon [0,1]^2 \to [0,1]$ we define its residual transform

$$\mathcal{R}[O](u,v) = \sup\{z \in [0,1] | O(u,z) \le v\}.$$

and its deresiduation

$$\bar{\mathcal{R}}[O](u,v) = \inf\{z \in [0,1] | O(u,z) \ge v\}.$$

Lemma

Every quasi-copula Q satisfies

$$\bar{\mathcal{R}}\big[\mathcal{R}[Q]\big] = \big(\bar{\mathcal{R}} \circ \mathcal{R}\big)[Q] = Q.$$

Novel construction of quasi-copulas

F. Durante, E. P. Klement, R. Mesiar, C. Sempi, *Conjunctors and their residual implicators: characterizations and construction methods*, Mediterranean Journal of Mathematics 4(3):343-356, 2007.

Theorem

If A, B are quasi-copulas then so is

$$\bar{\mathcal{R}}\Big[(1-\alpha)\mathcal{R}[A] + \alpha\mathcal{R}[B]\Big]$$

regardless of $\alpha \in [0, 1]$.

Question

If A and B are copulas, is the constructed operation also a copula ?

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Vertical mixture

Definition

- $\alpha \in [0,1]$
- $A, B\colon [0,1]^2 \to [0,1]$

Operation

$$A *_\alpha B = \bar{\mathcal{R}}\big[(1-\alpha)\mathcal{R}[A] + \alpha\mathcal{R}[B]\big]$$

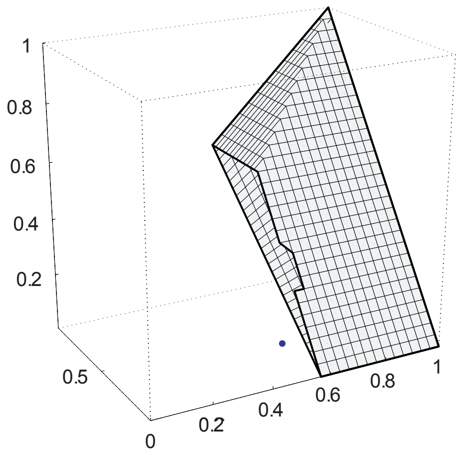
is the vertical α-mixture of A and B.

Properties

For quasi-copulas A and B

- $A *_0 B = A$ and $A *_1 B = B$
- $(A *_\alpha B)_{\alpha\in[0,1]}$
- $A *_\alpha B$ often violates commutativity even if A and B do not

Geometry of vertical mixtures



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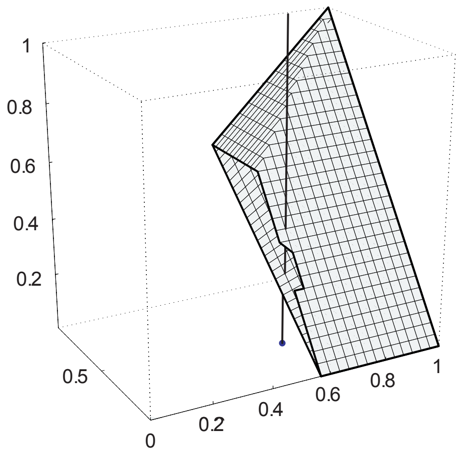

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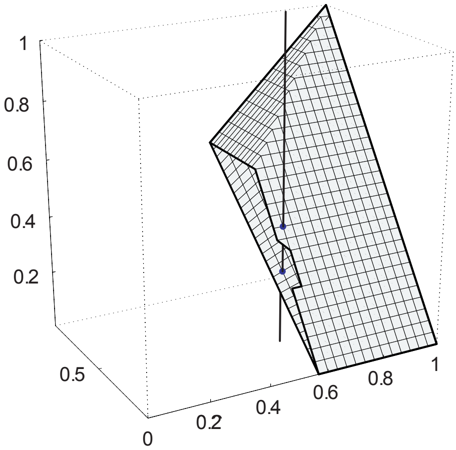

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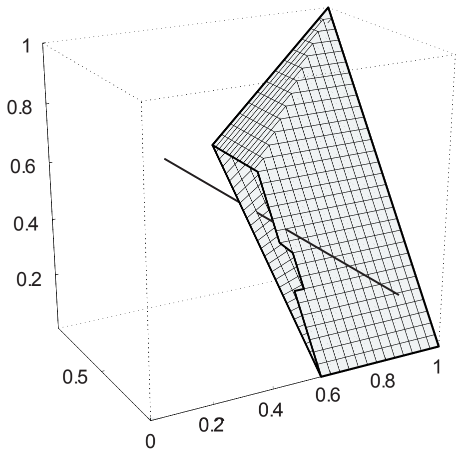

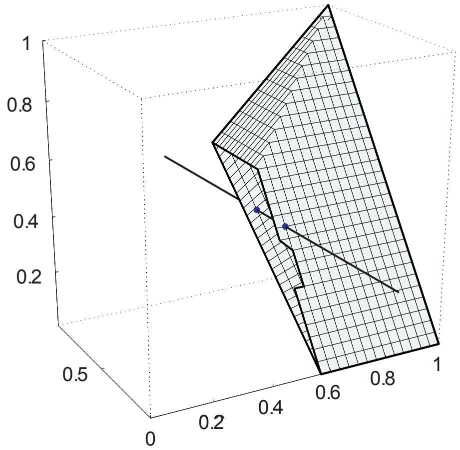
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0.8
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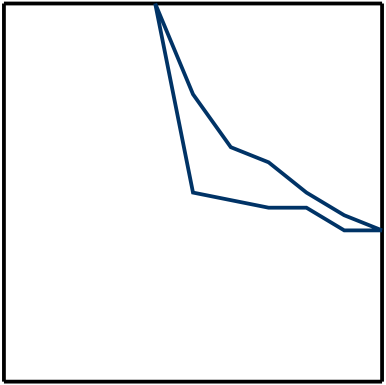
0.4
0.2
0.5
0
0.2
0.4
0.6
0.8
1



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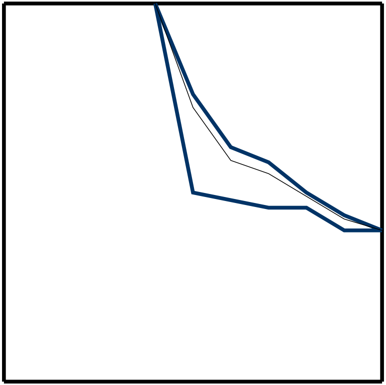

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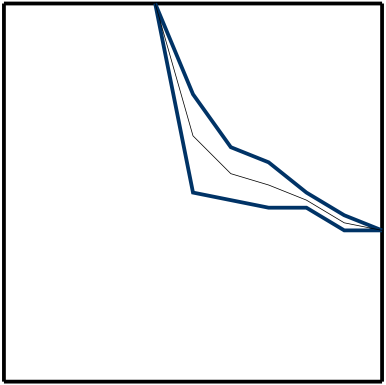

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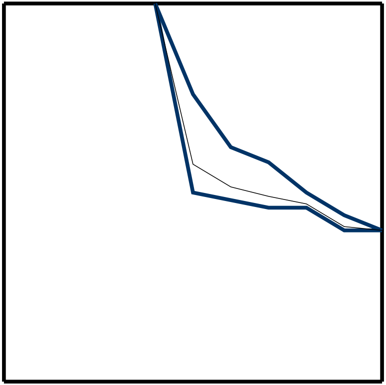

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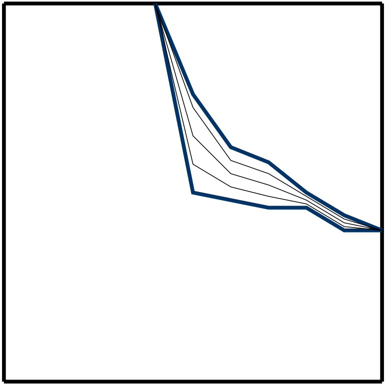
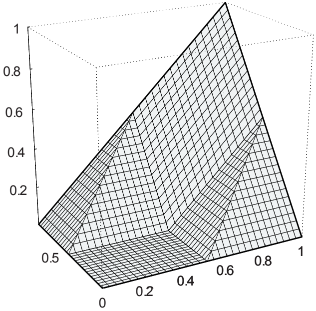
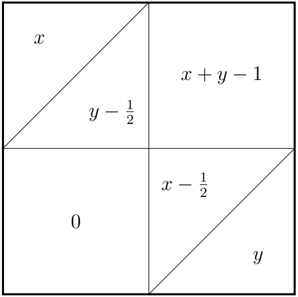


Example 1

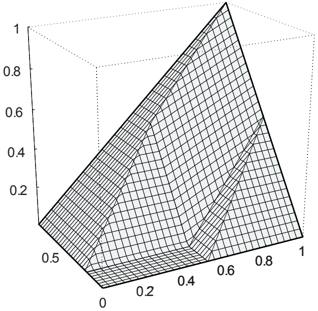
C

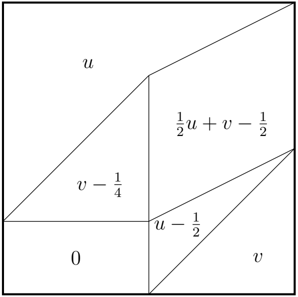




Example 1

$$C *_{0.5} M$$





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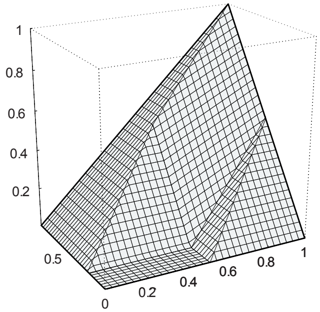
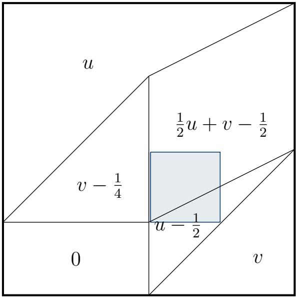




Example 2

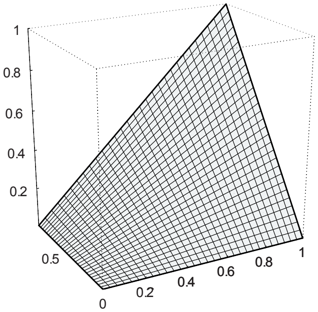
Put $C_\alpha = \Pi *_\alpha M$. Then

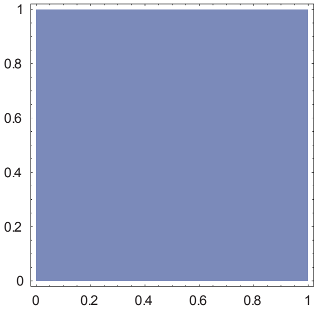
$$C_\alpha(u,v) = \min\left\{u, \frac{uv}{1-\alpha+\alpha u}\right\}$$

- C_α is a copula regardles of $\alpha \in [0,1]$
- up to the case $\alpha \in \{0,1\}$ the copula C_α is noncommutative
- the family $(C_\alpha)_{\alpha\in[0,1]}$ is increasing in α

Example 2

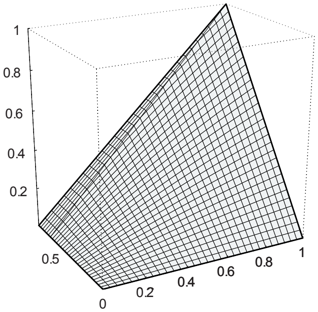
$$\Pi *_{0.0} M$$

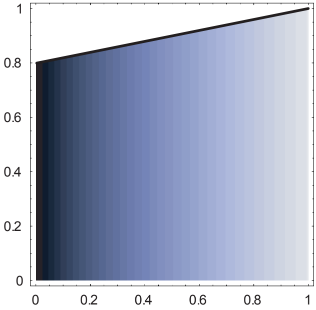




Example 2

$$\Pi *_{0.2} M$$





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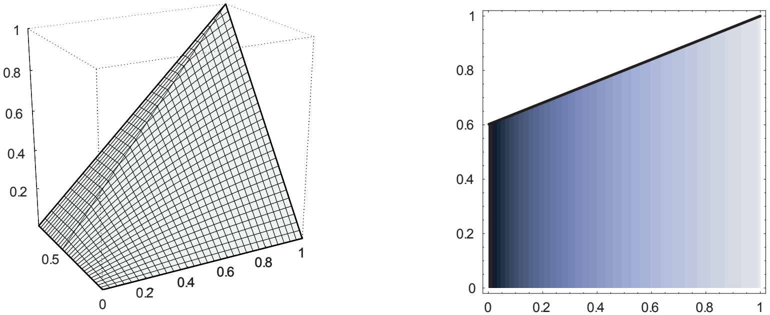
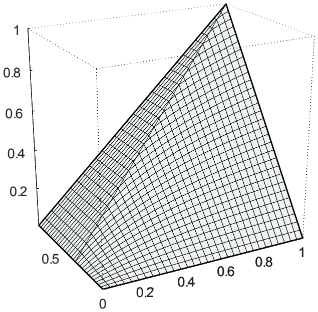
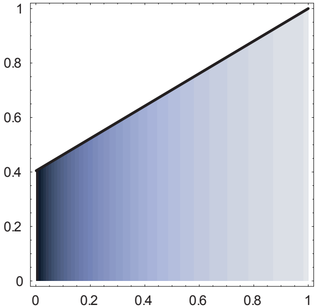
$$\Pi *_{0.4} M$$



Example 2

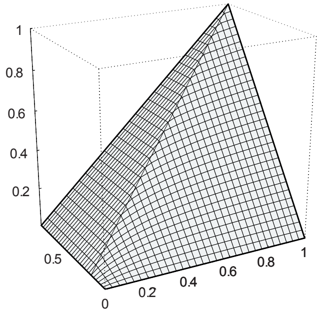
$$\Pi *_{0.6} M$$

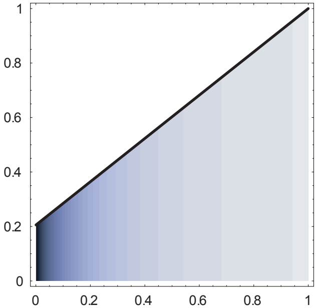




Example 2

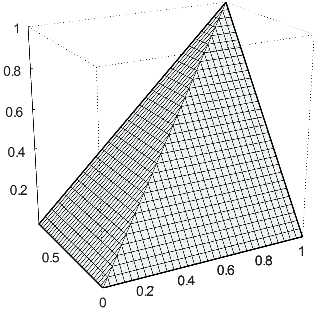
$$\Pi *_{0.8} M$$

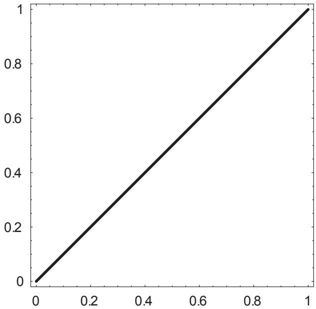




Example 2

$$\Pi *_{1.0} M$$





Example 3

Put $C_\alpha = M *_\alpha W$. Then

$$C_\alpha(u, v) = \mathrm{Glue}(\langle W, 0, \alpha\rangle, \langle M, \alpha, 1\rangle)$$

- C_α is a copula regardles of $\alpha \in [0, 1]$
- up to the case $\alpha \in \{0, 1\}$ the copula C_α is noncommutative
- the family $(C_\alpha)_{\alpha\in[0,1]}$ is decreasing in α
- every memenber of the family is singular

Differential tools

Convention

For $A\colon [0,1] \to [0,1]$ we denote by $\partial_1 A(u,v)$ $[\partial_2 A(u,v)]$ the value of the partial derivative of A along the first [the second] variable at the argument (u,v).

Lemma

A quasi-copula A is a copula iff the mapping $v \mapsto \partial_1 A(u,v)$ is nondecreasing for λ-almost every $u \in [0,1]$.

Note

Let a copula C be the distribution function of a random vector (U,V). Then

$$F_{V|U=u}(v) = P\left[V \leq v | U = u\right] = \partial_1 C(u,v).$$

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Characterisation of vertically mixable copulas

Theorem

A copula A is vertically α-mixable with a copula B iff the mappings

$$v \mapsto \partial_1 A(u,v) + \frac{\alpha \partial_1 A_B(u,v) \partial_2 A(u,v)}{1 - \alpha + \alpha \partial_2 A_B(u,v)}$$

where

$$A_B(u,v) = \sup\{\, z \in [0,1] \,|\, A(u,v) = B(u,z) \}$$

are nondecreasing for almost all $u \in [0,1]$.

Example

- $A_M = A(u,v)$
- $A_\Pi = \frac{A(u,v)}{u}$
- $A_W = 1 - u + A(u,v)$

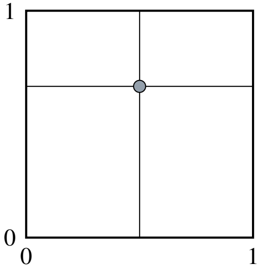
Vertical mixatures with M

Corollary

Let $\alpha \in \,]0,1[$. A copula A is vertically α-mixable with M iff the mappings

$$v \mapsto \frac{\partial_1 A(u,v)}{1 + \frac{\alpha}{1-\alpha}\partial_2 A(u,v)}$$

are nondecreasing for almost all $u \in [0,1]$.



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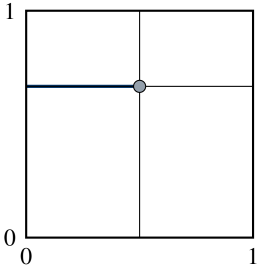

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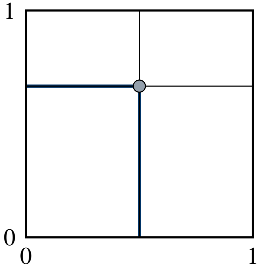

Vertical mixtures with M (cont'd)

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Corollary

Every copula with convex vertical sections is vertically mixable with M. In particullar every stochastically increasing copula is vertically mixable with M.

Is there a probabilistic interpretation?

Theorem (folklore)

Let X, Y be random variables (defined on a common probability space) and $\alpha \in [0, 1]$. If X and Y are totally increasingly dependent, then

$$Q_{\alpha X+(1-\alpha)Y} = \alpha Q_X + (1-\alpha) Q_Y.$$

Another folkloric issue

Let X, Y, Z be three random variables distributed uniformly over the unit interval. If there exists $\alpha \in]0, 1[$ for which the joint distribution function of $(X, \alpha Y + (1-\alpha)Z)$ is a copula, then $Y =_P Z$.

Question

Which operations on random vectors do correspond to vertical mixtures of copulas?

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